On various applications of message passing on factor graphs

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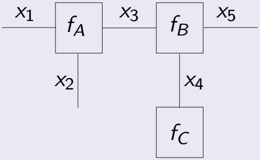
Two threads

1. Particular problem of carrier-phase synchronization in single-carrier communications systems.
2. Message-passing algorithms for various applications.

Forney-style factor graphs (FFGs)

- Factor graphs represent the factorization of a function.
- Example

$$f(x_1, x_2, x_3, x_4, x_5) = f_A(x_1, x_2, x_3) f_B(x_3, x_4, x_5) f_C(x_4).$$



- Rules for drawing a factor graph
  - A node for every factor
  - An edge for every variable
  - Node g is connected to edge x iff variable x appears in factor g

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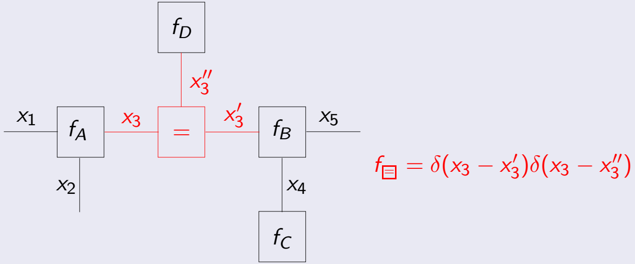

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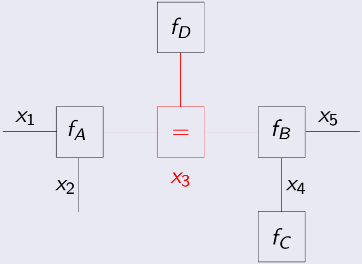

$$f_{\boxed{=}} = \delta(x_3 - x_3')\delta(x_3 - x_3'')$$

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Computing marginals

- Given: Discrete probability mass function

$$f(x_1, \ldots, x_8) = \left(f_1(x_1) f_2(x_2) f_3(x_1, x_2, x_3, x_4)\right) \cdot \Big(f_4(x_4, x_5, x_6) f_5(x_5) \left(f_6(x_6, x_7, x_8) f_7(x_7)\right)\Big)$$

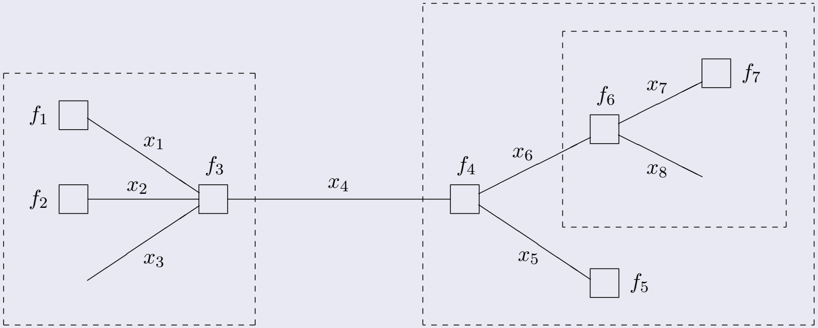
- Wanted: Marginal probability

$$p(x_4) = \sum_{x_1, x_2, x_3, x_5, x_6, x_7, x_8} f(x_1, \ldots, x_8)$$

- This factorization can be represented by a factor graph.

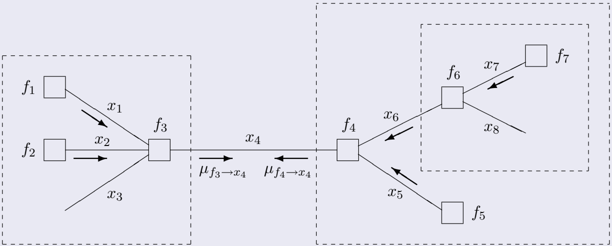
Computing marginals

$$
\begin{aligned}
f(x_1, \ldots, x_8) \quad = \quad & \big(f_1(x_1) f_2(x_2) f_3(x_1, x_2, x_3, x_4)\big) \cdot \\
& \Big(f_4(x_4, x_5, x_6) f_5(x_5) \big(f_6(x_6, x_7, x_8) f_7(x_7)\big)\Big)
\end{aligned}
$$



Computing marginals

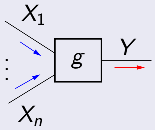
$$p(x_4) = \underbrace{\left(\sum_{x_1}\sum_{x_2}\sum_{x_3} f_3(x_1,x_2,x_3,x_4)f_1(x_1)f_2(x_2)\right)}_{\mu_{f_3\to x_4}} \cdot \underbrace{\left(\sum_{x_5}\sum_{x_6} f_4(x_4,x_5,x_6)f_5(x_5)\underbrace{\left(\sum_{x_7}\sum_{x_8} f_6(x_6,x_7,x_8)f_7(x_7)\right)}_{\mu_{f_6\to x_6}}\right)}_{\mu_{f_4\to x_4}}$$



Sum-product algorithm ("belief propagation")

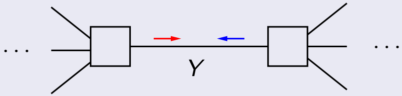
Sum-product rule

$$\mu(y) \quad \propto \quad \sum_{x_1,\ldots,x_n} g(x_1,\ldots,x_n,y)\cdot\mu(x_1)\ldots\cdot\mu(x_n).$$



Marginal

$$p(y) \propto \mu_{\rightarrow}(y)\mu_{\leftarrow}(y)$$



Cyclic graphs

Still applicable, but approximate marginals; may not convergence!

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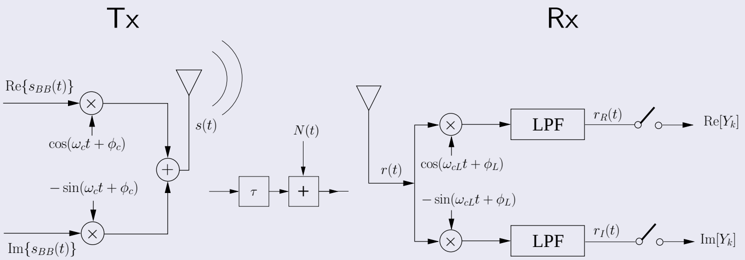
Single-Carrier Communications System

Block diagram



$s_{BB}(t) = \sum_i c_i g(t - iT)$

$r(t) = s(t - \tau) + N(t)$

$s(t) = \mathrm{Re}\left\{ s_{BB}(t) e^{j(\omega_c t + \phi_c)} \right\}$

τ: channel delay	ω_c: carrier frequency	ϕ_c: carrier phase
T: symbol length	$g(t)$: pulse shape	c_i: data symbols
$s_{BB}(t)$: baseband signal	$s(t)$: passband signal	$N(t)$: AWGN

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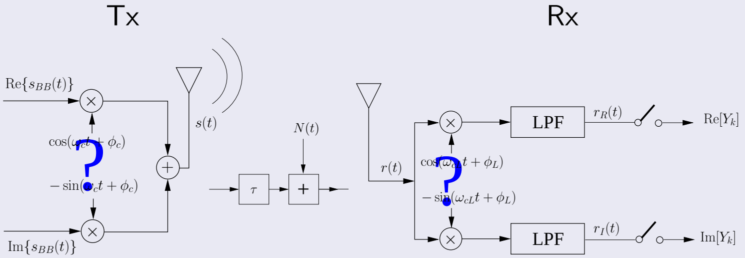

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Questions

Modeling

Which physical mechanisms are responsible for (phase) noise?
How can (phase) noise be modeled?

Algorithms

How can phase-estimation algorithms systematically be derived?

Performance limits

How well can the (noisy) carrier phase be estimated?
How much does the information rate decrease due to phase noise?

Contributions

Modeling

Which physical mechanisms are responsible for (phase) noise?
How can (phase) noise be modeled?

Simple intuitive model for phase noise.

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As message passing on factor graph of the system a hand.

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Computation of Cramér-Rao bounds/information rates/capacities.

Channel model

Model for single-carrier system with slowly varying phase offset

$$Y_k = X_k e^{j\Theta_k} + W_k, \quad W_k \sim \mathcal{CN}_{0,\sigma_N^2}.$$

Constant-phase model

$$\Theta_k = \Theta \in [0, 2\pi).$$

Random-walk phase model

$$\Theta_k = (\Theta_{k-1} + N_k) \bmod 2\pi, \quad N_k \sim \mathcal{N}_{0,\sigma_N^2}.$$

σ_N^2 and σ_W^2 are assumed to be known.
The input symbols X_k are protected by an error-correcting code.

Contributions

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Algorithms for joint decoding and phase estimation

Estimation task

Given a block of observations $Y \triangleq (Y_1, Y_2, \ldots, Y_N)$, infer:

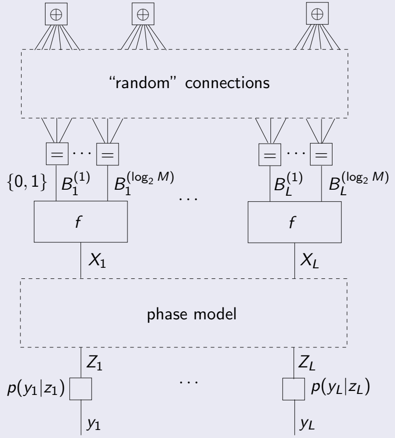
- the coded symbols $X \triangleq (X_1, X_2, \ldots, X_N)$
- the phase $\Theta \triangleq (\Theta_1, \Theta_2, \ldots, \Theta_N)$.

Derivation of message-passing estimation algorithms [Wiberg, 1996]

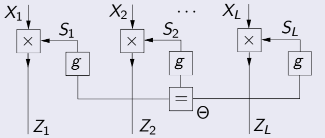
1. Draw factor graph of joint pdf $p(x, y, \theta)$.
2. Apply sum-product rule at each node.
3. If sum-product rule is infeasible at a certain node, then apply an approximation $=$ choose appropriate message types.
4. Choose an update schedule.

Factor graphs

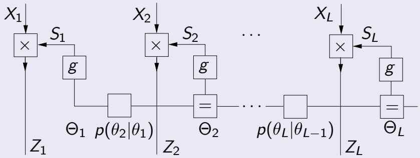
Factor graph of $p(x, y, \theta)$



Constant-phase model



Random-walk phase model



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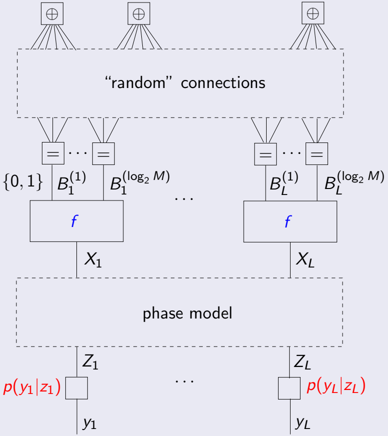

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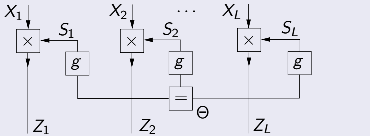

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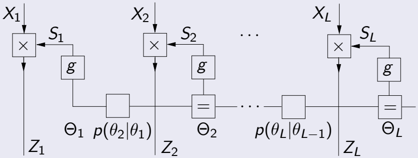

$$p(y_k|z_k) \triangleq (2\pi\sigma_N^2)^{-1}\ e^{-|y_k - z_k|^2/2\sigma_N^2} \qquad \delta_f(\cdot) \triangleq \delta\left[f\left(b_k^{(1)}, \ldots, b_k^{(\log_2 M)}\right) - x_k\right]$$

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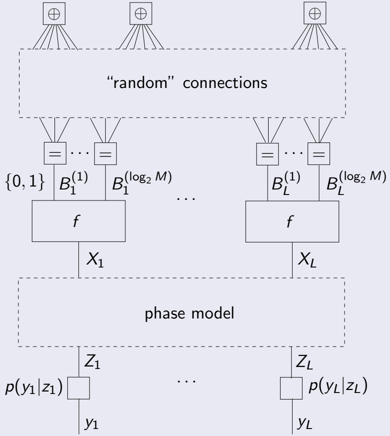

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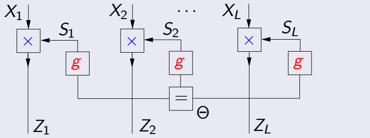

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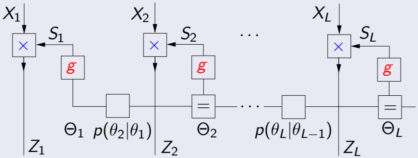

$S_k \triangleq e^{j\Theta_k}$ $\quad g(\theta_k, s_k) \triangleq \delta(s_k - e^{j\theta_k})$ $\quad Z_k \triangleq X_k S_k$ $\quad f_{\boxtimes}(x_k, s_k, z_k) \triangleq \delta(z_k - x_k s_k)$

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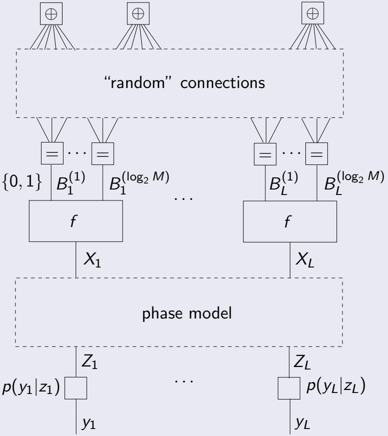

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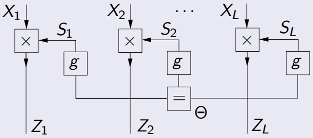

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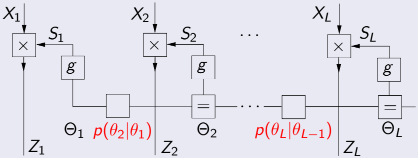
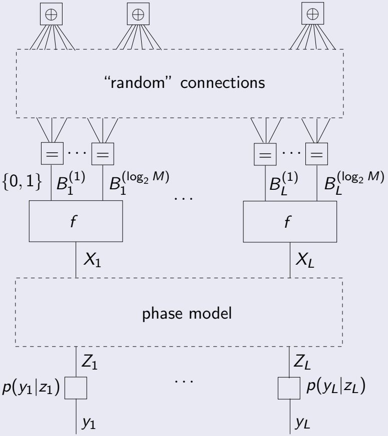


$$p(\theta_k|\theta_{k-1}) \triangleq (2\pi\sigma_W^2)^{-1/2} \sum_{n\in\mathbb{Z}} e^{-((\theta_k-\theta_{k-1})+n2\pi)^2/2\sigma_W^2}$$

Sum-product rule

Factor graph of $p(x, y, \theta)$



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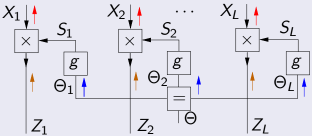

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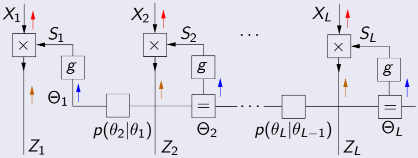
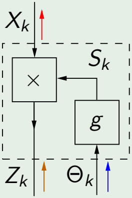


Sum-product rule

Example

Sum-product rule

$$\begin{aligned}\mu(x_k) &\propto \int_0^{2\pi}\int_{z_k} \delta(z_k - x_k e^{j\theta_k})\mu(\theta_k)\mu(z_k)\, d\theta_k\, dz_k, \\ &\propto \int_0^{2\pi} \mu(\theta_k)\mu\left(x_k e^{j\theta_k}\right)\, d\theta_k, \\ &\propto \int_0^{2\pi} \mu(\theta_k) e^{-|x_k e^{j\theta_k} - y_k|^2/2\sigma_W^2}\, d\theta_k\end{aligned}$$



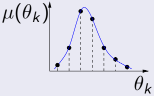
Intractable integral!

Message Types

$$\mu(x_k) \quad \propto \quad \int_0^{2\pi} \mu(\theta_k) e^{-|x_k e^{j\theta_k} - y_k|^2/2\sigma_W^2} \, d\theta_k$$

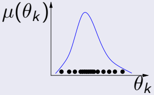
Numerical integration

$$\mu(x_k) \quad \propto \quad \sum_i \mu(\hat{\theta}_k^{(i)}) e^{-|x_k e^{j\hat{\theta}_k^{(i)}} - y_k|^2/2\sigma_W^2}$$



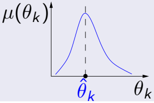
Particle method

$$\mu(x_k) \quad \propto \quad \sum_i e^{-|x_k e^{j\hat{\theta}_k^{(i)}} - y_k|^2/2\sigma_W^2}$$



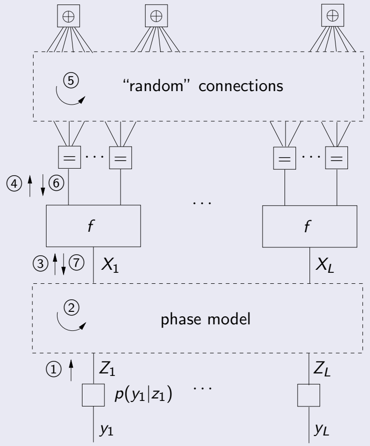
Decision based

$$\mu(x_k) \approx e^{-|x_k e^{j\hat{\theta}_k} - y_k|^2/2\sigma_W^2}$$

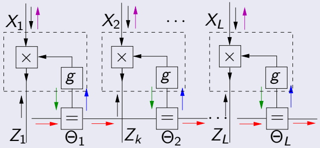


Scheduling

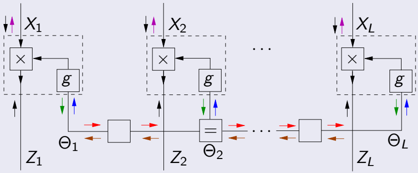
Factor graph of $p(x, y, \theta)$



Constant-phase model



Random-walk phase model



Unification

Particle methods

Importance Sampling, Markov-Chain Monte Carlo, Metropolis-Hastings Algorithm, Gibbs Sampling, Simulated Annealing, Particle Filtering

Decision based

Iterative Conditional Modes, Gradient Methods, Stochastic Approximation, Expectation maximization, SAGE, Gradient EM, Natural-Gradient Methods, Backpropagation Algorithm

Combinations

Monte-Carlo EM, Stochastic EM

Interpretation as message passing on factor graphs

Identified generic local message-update rules for each approach.

Joint work with Sascha Korl

Why we care . . .

Divide and conquer

Global estimation/detection problem accomplished by simple local computations. Complicated mathematical derivations avoided.

Disciplined approach

Deriving novel algorithms systematically by listing possible message update rules at each node in the graph.

Mish mash

Straightforward to combine several approaches, e.g., decision-based, particle-based etc., in one single algorithm.

Plug and play

Deriving novel algorithms by combining tabulated message update rules. Efficient use of earlier work.

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Computation of Cramér-Rao bounds/information rates/capacities.

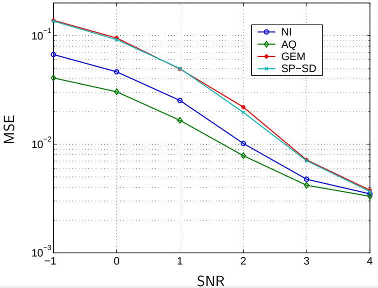
Cramér-Rao-type bounds

What?

Lower bounds on the mean-squared estimation error (MSE)

e.g., $\mathrm{MSE} = \mathrm{E}_{Y\theta}\left[(\theta - \hat{\theta}(Y))^2\right]$.

Motivation



Assessment of practical estimators

e.g., phase-estimation algorithms

Cramér-Rao-type bounds

Three different types

- Standard Cramér-Rao bounds: parameters
- Bayesian Cramér-Rao bounds: random variables
- Hybrid Cramér-Rao bounds: parameters and random variables

Bayesian Cramér-Rao bound: scalar case

Theorem (Bayesian Cramér-Rao bound)

Let $p(x, y)$ be the joint pdf of $x \in \mathbb{R}$ and $y \triangleq (y_1, \ldots, y_N)$.
If $p(x)$ is zero at boundary of its support, then for any regular $\hat{x}(y)$:

$$E_{XY}\left[(x - \hat{x}(y))^2\right] \geq J^{-1},$$

where the Bayesian information matrix J is defined as:

$$J \triangleq E_{XY}\left[\left(\frac{\partial}{\partial x} \log p(x, y)\right)^2\right].$$

Properties

- MAP-estimator achieves bound as SNR or $N \rightarrow \infty$.
- BCRB holds for any regular $\hat{x}(y)$ as SNR or $N \rightarrow \infty$.

Bayesian Cramér-Rao bound: simple example

Example (Mean of a Gaussian random variable)

$Y = X + Z$ with $Z \sim \mathcal{N}_{(0,\sigma^2)}$ with σ^2 known and $X \in \mathbb{R}$ unknown.

Estimate X from observations $y_1, y_2, \ldots, y_N$ with prior $p(X)$ for X.

$$p(x, y_1, y_2, \ldots, y_N) = p(x) \prod_{k=1}^{N} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-y_k)^2/2\sigma^2}.$$

$$\begin{aligned} \mathbf{J} &= -\mathrm{E}_{XY}\left[\frac{d^2}{dx^2}\log p(X,Y)\right] \\ &= \frac{N}{\sigma^2} - \mathrm{E}_X\left[\frac{d^2}{dx^2}\log p(X)\right]. \end{aligned}$$

$$\mathrm{E}_{XY}[(\hat{x}(X) - X)^2] \geq \mathbf{J}^{-1} = \left(\frac{N}{\sigma^2} - \mathrm{E}_X\left[\frac{d^2}{dx^2}\log p(X)\right]\right)^{-1}$$

If $p(x)$ is Gaussian, then BCRB = minimum achievable MSE!

Vector case

Bayesian Cramér-Rao bound for component X_k

Given: joint pdf $p(x, y)$ of $x \triangleq (x_1, \ldots, x_M)$ and $y \triangleq (y_1, \ldots, y_N)$.

Lower bound for the MSE $\mathrm{E}_{X_k Y}\left[(X_k - \hat{x}_k(Y))^2\right]$?

From marginal

$$\mathrm{E}_{X_k Y}\left[(X_k - \hat{x}_k(Y))^2\right] \geq J_k^{-1},$$

with $J_k \triangleq \mathrm{E}_{X_k Y}\left[\left(\frac{\partial}{\partial x_k} \log p(x_k, y)\right)^2\right]$.

From joint pdf

$$\mathrm{E}_{X_k Y}\left[(X_k - \hat{x}_k(Y))^2\right] \geq \left[J^{-1}\right]_{kk},$$

with $J_{ij} \triangleq \mathrm{E}_{XY}\left[\frac{\partial}{\partial x_i} \log p(x, y) \left(\frac{\partial}{\partial x_j} \log p(x, y)\right)^T\right]$.

BCRB from marginal is tighter than from joint pdf, but more difficult to compute.

Algorithms

From joint pdf

- J is often sparse.
- Only diagonal elements of inverse required.
- Local computation of "small" matrices (message passing).

From marginal

- J_k is usually dense.
- Key to J_k: $\frac{\partial}{\partial x_k} \log p(x_k, y) = \mathrm{E}_{X_{\sim k}|X_k Y}\left[\frac{\partial}{\partial x_k} \log p(X, Y)\right]$
- Expectation computed by sum-product algorithm
  (or "belief propagation" or "probability propagation").

Algorithms

Overview

We propose efficient and simple message-passing algorithms:

- for computing standard, Bayesian, hybrid Cramér-Rao bounds
- following both strategies.

Two examples

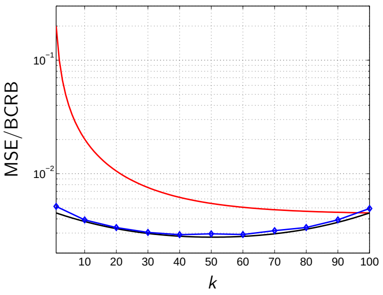
1. Random-walk phase model: Bayesian CRB from joint pdf
2. AR model: standard CRB from marginal.

Example 1: phase estimation; Bayesian CRB from joint pdf

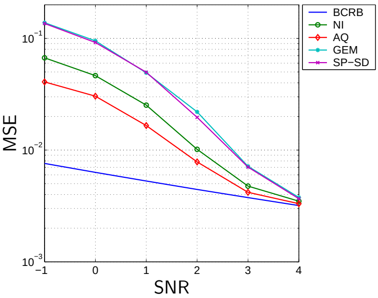
Infer $X = X_1, \ldots, X_L$ and $\Theta = \Theta_1, \ldots, \Theta_L$ from $Y = Y_1, \ldots, Y_L$

Random-walk phase model with $\sigma_W^2 = 10^{-4}\text{rad}^2$ with $L = 100$

MSE/BCRB for Θ_k (SNR $=$ 4dB)



Average MSE/BCRB



Example 2: AR model; standard CRB from marginal

Example (AR model)

Let $X_1, X_2, \ldots$ be a real random process defined by:

$$X_k = a_1 X_{k-1} + a_2 X_{k-2} + \cdots + a_M X_{k-M} + U_k, \quad U_k \overset{\text{i.i.d.}}{\sim} \mathcal{N}_{(0,\sigma_U^2)}$$

and let the process $Y = Y_1, Y_2, \ldots$ be defined as:

$$Y_k = X_k + W_k, \quad U_k \overset{\text{i.i.d.}}{\sim} \mathcal{N}_{(0,\sigma_W^2)}.$$

Task

Estimate $a_1, \ldots, a_M$, σ_U^2, σ_W^2, and X from observation Y.

Example 2: AR model; standard CRB from marginal

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Task

Estimate $a_1, \ldots, a_M$, σ_U^2, σ_W^2, and X from observation Y.

Observation

ML/MAP/MMSE-estimators are infeasible!
Neither can their MSE be determined $\Rightarrow$ lower bounds on MMSE.

Standard Cramér-Rao bound

Theorem (Standard Cramér-Rao bound)

Let $p(y;\theta)$ be the pdf of $y \triangleq (y_1, \ldots, y_N)$ parameterized by $\theta \triangleq (\theta_1, \ldots, \theta_N)$. If $\hat{x}(y)$ is regular and unbiased estimator, then

$$E_{Y;\theta}\left[(\theta - \hat{\theta}(y))(\theta - \hat{\theta}(y))^T\right] \geq F^{-1}(\theta),$$

where the Fisher information matrix $F(\theta)$ is defined as

$$F(\theta) \triangleq E_{Y;\theta}\left[\nabla_\theta \log p(y;\theta) \nabla_\theta^T \log p(y;\theta)\right].$$

Properties

- ML-estimator achieves bound as SNR or $N \to \infty$.
- Standard CRB holds for any regular $\hat{\theta}(y)$ as SNR or $N \to \infty$.

Standard Cramér-Rao bound for the AR model

Example (AR model)

Let $X_1, X_2, \ldots$ be a real random process defined by:

$$X_k = a_1 X_{k-1} + a_2 X_{k-2} + \cdots + a_M X_{k-M} + U_k, \quad U_k \overset{\text{i.i.d.}}{\sim} \mathcal{N}_{(0,\sigma_U^2)}$$

and let the process $Y = Y_1, Y_2, \ldots$ be defined as:

$$Y_k = X_k + W_k, \quad U_k \overset{\text{i.i.d.}}{\sim} \mathcal{N}_{(0,\sigma_W^2)}.$$

Ingredients for the Cramér-Rao bound

$\theta = (\mathbf{a}, \sigma_U^2, \sigma_W^2)$

$F(\theta) \triangleq \mathrm{E}_{Y;\theta}\left[\nabla_\theta \log p(Y;\theta) \nabla_\theta^T \log p(Y;\theta)\right]$

$p(y;\theta) \triangleq \int_x p(x,y;\theta)$

$\nabla_\theta \log p(y;\theta) = \mathrm{E}_{X|\theta,y}\left[\nabla_\theta \log p(X,y;\theta)\right].$

Expectations $\nabla_\theta \log p(y;\theta) = \mathrm{E}_{X|\theta y}\left[\nabla_\theta \log p(X,y;\theta)\right]$

$$p(x,y|\mathbf{a},\sigma_W^2,\sigma_U^2) = \prod_k \underbrace{\mathcal{N}\left(x_k - \sum_{n=1}^{M} a_n x_{k-n} \,|\, 0, \sigma_U^2\right)}_{f_1(x_k,\ldots,x_{k-M},\mathbf{a},\sigma_U^2)} \underbrace{\mathcal{N}\left(y_k - x_k \,|\, 0, \sigma_W^2\right)}_{f_2(x_k,\sigma_W^2,y_k)}.$$

As a consequence:

$$\nabla_\theta \log p(x,y|\mathbf{a},\sigma_W^2,\sigma_U^2) = \sum_k \nabla_\theta \log f_1(x_k,\ldots,x_{k-M},\mathbf{a},\sigma_U^2) + \sum_k \nabla_\theta \log f_2(x_k,\sigma_W^2,y_k).$$

Expectations $\nabla_\theta \log p(y;\theta) = \mathrm{E}_{X|\theta y}\left[\nabla_\theta \log p(X,y;\theta)\right]$ (2)

$$\mathrm{E}_{X|\mathbf{a}\,\sigma_W^2\sigma_U^2\,Y}\left[\nabla_{a_i}\log f_1(X_k,\ldots,X_{k-M},\mathbf{a},\sigma_U^2)\right]$$

$$= \frac{1}{\sigma_U^2}\left(\mathrm{E}_{X|\mathbf{a}\,\sigma_W^2\sigma_U^2\,Y}\left[X_{k-i}X_k\right] - \sum_{\ell=1}^{M} a_\ell \mathrm{E}_{X|\mathbf{a}\,\sigma_W^2\sigma_U^2\,Y}\left[X_{k-i}X_{k-\ell}\right]\right)$$

$$\mathrm{E}_{X|\mathbf{a}\,\sigma_W^2\sigma_U^2\,Y}\left[\nabla_{\sigma_U^2}\log f_1(X_k,\ldots,X_{k-M},\mathbf{a},\sigma_U^2)\right]$$

$$= -\frac{1}{2\sigma_U^2} + \frac{1}{2\sigma_U^4}\left(\mathrm{E}_{X|\mathbf{a}\,\sigma_W^2\sigma_U^2\,Y}\left[X_k^2\right] - 2\sum_{\ell=1}^{M} a_\ell \mathrm{E}_{X|\mathbf{a}\,\sigma_W^2\sigma_U^2\,Y}\left[X_kX_{k-\ell}\right]\right.$$

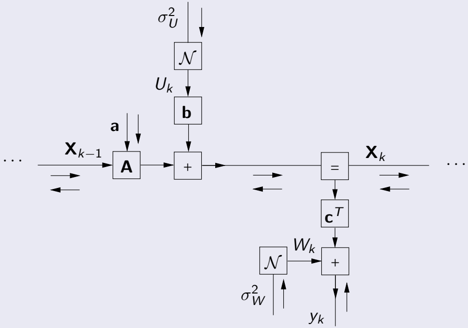
$$\left. + \sum_{\ell=1}^{M}\sum_{m=1}^{M} a_\ell a_m \mathrm{E}_{X|\mathbf{a}\,\sigma_W^2\sigma_U^2\,Y}\left[X_{k-\ell}X_{k-m}\right]\right)$$

$$\mathrm{E}_{X|\mathbf{a}\,\sigma_W^2\sigma_U^2\,Y}\left[\nabla_{\sigma_W^2}\log f_2(X_k,\sigma_W^2,y_k)\right]$$

$$= -\frac{1}{2\sigma_W^2} + \frac{1}{2\sigma_W^4}\left(y_k^2 - 2y_k \mathrm{E}_{X|\mathbf{a}\,\sigma_W^2\sigma_U^2\,Y}\left[X_k\right] + \mathrm{E}_{X|\mathbf{a}\,\sigma_W^2\sigma_U^2\,Y}\left[X_k^2\right]\right).$$

Expectations $\nabla_\theta \log p(y;\theta) = \mathrm{E}_{X|\theta y}\left[\nabla_\theta \log p(X, y; \theta)\right]$ (3)

Computation of means/variances/correlations



$\mathrm{E}_{X|\mathbf{a}\,\sigma_W^2\sigma_U^2\,Y}$ computed by forward/backward Kalman recursions.

(= sum-product rule with Gaussian messages)

Computing the Fisher information matrix of AR model

1. Generate a list of samples $\{\hat{y}^{(j)}\}_{j=1}^{N}$ from $p(y|\theta)$.
2. For $j = 1, \ldots, N$:
   - Forward and backward Kalman recursion with $y = \hat{y}^{(j)}$.
   - Evaluate the expression:

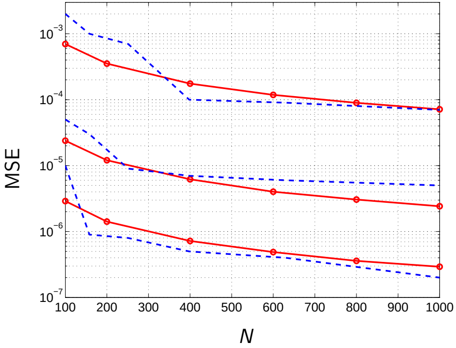
$$\mathrm{E}_{X|\theta,\hat{y}^{(j)}}\left[\nabla_\theta \log p(X, \hat{y}^{(j)}; \theta)\right].$$

3. Compute the estimate $\hat{\mathbf{F}}(\theta)$ for $\mathbf{F}(\theta)$:

$$\hat{\mathbf{F}}(\theta) \triangleq \frac{1}{N}\sum_{j=1}^{N}\Big[\mathrm{E}_{X|\theta,\hat{y}^{(j)}}[\nabla_\theta \log p(X, \hat{y}^{(j)}; \theta)] \\ \mathrm{E}^{T}_{X|\theta,\hat{y}^{(j)}}[\nabla_\theta \log p(X, \hat{y}^{(j)}; \theta)]\Big].$$

Results

Results for σ_W^2 ($\sigma_U^2 = 0.1$; σ_W^2=0.001,0.01,0.1)



Estimation algorithm by Sascha Korl.

Summary: Cramér-Rao-type bounds

What?

Lower bounds on the mean-squared estimation error (MSE)

Three different types

- Standard Cramér-Rao bounds: parameters
- Bayesian Cramér-Rao bounds: random variables
- Hybrid Cramér-Rao bounds: parameters and random variables.

Two strategies

Inverse of information matrix of joint pdf or marginal.

Algorithms

We propose message-passing algorithms for computing the three types of CRBs following both strategies.

Other applications

- Other types of bounds,
  e.g., Weiss-Weinstein (discrete variables), Bhattacharyya, etc.
- Information geometry: natural-gradient-based algorithms
- Machine learning: computation of Fisher kernels.

Contributions

Modeling

Which physical mechanisms are responsible for (phase) noise?
How can (phase) noise be modeled?

Simple intuitive model for phase noise.

Algorithms

How can phase-estimation algorithms systematically be derived?

As message passing on factor graph of the system a hand.

Performance limits

How well can the (noisy) carrier phase be estimated?
How much does the information rate decrease due to phase noise?

Computation of Cramér-Rao bounds/information rates/capacities.

Information rate: introduction

Objective

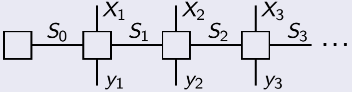
Information rate $I(X;Y) \triangleq \lim_{n\to\infty} \frac{1}{n} I(X_1,\ldots,X_n;Y_1,\ldots,Y_n)$ between input process $X = (X_1, X_2, \ldots)$ and output process $Y = (Y_1, Y_2, \ldots)$ of time-invariant discrete-time channel with memory.

State-space representation

An ergodic stochastic process $S = (S_0, S_1, S_2, \ldots)$ such that

$$p(x^n, y^n, s_0^n) = p(s_0) \prod_{k=1}^{n} p(x_k, y_k, s_k | s_{k-1})$$

for all $n > 0$ and with $p(x_k, y_k, s_k | s_{k-1})$ not depending on k.



Basic principle

Reminder

$$I(X;Y) = h(Y) - h(Y|X).$$

Shannon-McMillan-Breiman theorem

- $-\frac{1}{n}\log p(X^n) \to H(X)$ w.p.1
- $-\frac{1}{n}\log p(Y^n) \to h(Y)$ w.p.1
- $-\frac{1}{n}\log p(X^n, Y^n) \to H(X) + h(Y|X)$ w.p.1.

Notation: $X^n \triangleq (X_1, \ldots, X_n)$ and $Y^n \triangleq (Y_1, \ldots, Y_n)$

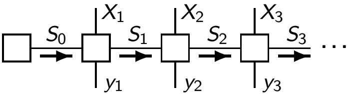
Algorithm

1. Sample two "very long" sequences x^n and y^n.
2. Compute $\log p(x^n)$, $\log p(y^n)$, and $\log p(x^n, y^n)$.
3. $\hat{I}(X;Y) \triangleq \frac{1}{n}\log p(x^n, y^n) - \frac{1}{n}\log p(x^n) - \frac{1}{n}\log p(y^n)$.

[Arnold et al., Pfister et al., Sharma et al.]

Basic principle

Compute $\log p(x^n)$, $\log p(y^n)$, and $\log p(x^n, y^n)$.

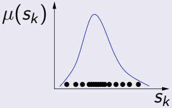


Discrete input space $\mathcal{X}$ and state-space $\mathcal{S}$ [e.g., Arnold et al.]

Forward sum-product sweep = forward BCJR-recursion

Continuous input space $\mathcal{X}$ and state-space $\mathcal{S}$

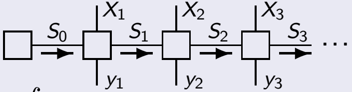
Forward sum-product sweep by particle filtering.
Expression $p(x_k, s_k|s_{k-1})$ not required!
E.g., stochastic differential/difference equation.



Forward sum-product sweep

Computation of $p(y^n) \triangleq \int_{x^n} \int_{s_0^n} p(x^n, y^n, s_0^n)$.

Recursion



$$\begin{aligned}\mu_k(s_k) &= \int_{x_k} \int_{s_{k-1}} \mu_{k-1}(s_{k-1})\, p(x_k, y_k, s_k | s_{k-1}) dx_k\, ds_{k-1} \\ &= \int_{x^k} \int_{s_0^{k-1}} p(x^k, y^k, s^k) dx^k\, ds_0^{k-1}\end{aligned}$$

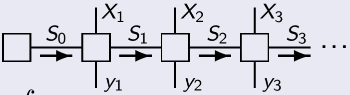
Marginal computed from message

$$p(y^n) = \int_{s_n} \mu_n(s_n).$$

Forward sum-product sweep

Computation of $p(y^n) \triangleq \int_{x^n} \int_{s_0^n} p(x^n, y^n, s_0^n)$.

Recursion with normalization



$$\mu_k(s_k) = \lambda_k \int_{x_k} \int_{s_{k-1}} \mu_{k-1}(s_{k-1})\, p(x_k, y_k, s_k | s_{k-1}) dx_k ds_{k-1}.$$

$$\int_{s_k} \mu_k(s_k) = 1 \text{ for all } k.$$

Marginal computed from normalization factors

$$\frac{1}{n} \sum_{k=1}^{n} \log \lambda_k = -\frac{1}{n} \log p(y^n).$$

Computation of normalization factors

Discrete input space $\mathcal{X}$ and state-space $\mathcal{S}$

Forward sum-product recursion = forward BCJR recursion

$$\lambda_k^{-1} = \sum_{s_k} \sum_{x_k} \sum_{s_{k-1}} \mu_{k-1}(s_{k-1})\, p(x_k, s_k | s_{k-1})\, p(y_k | x_k, s_k, s_{k-1}).$$

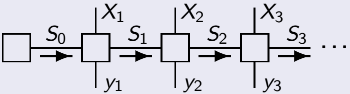
Continuous input space $\mathcal{X}$ and state-space $\mathcal{S}$

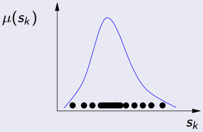
Forward sum-product recursion = particle filtering

$$\begin{aligned}
\lambda_k^{-1} &= \int_{s_k} \int_{x_k} \int_{s_{k-1}} \mu_{k-1}(s_{k-1})\, p(x_k, s_k | s_{k-1})\, p(y_k | x_k, s_k, s_{k-1}) ds_k dx_k ds_{k-1} \\
&= \mathrm{E}_{s_{k-1} s_k x_k}[p(y_k | x_k, s_k, s_{k-1})] \\
&\approx \frac{1}{N} \sum_{\ell=1}^{N} p_{Y_k | X_k, S_k, S_{k-1}}(y_k | \hat{x}_{k,\ell}, \hat{s}_{k,\ell}, \hat{s}_{k-1,\ell}).
\end{aligned}$$

Particle method

Algorithm

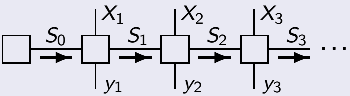


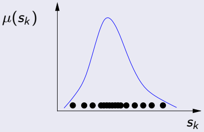


1. Begin with list $\{\hat{s}_{k-1,\ell}\}_{\ell=1}^{N}$ that represents μ_{k-1}.
2. Extend each particle $\hat{s}_{k-1,\ell}$ to three-tuple $(\hat{s}_{k-1,\ell}, \hat{x}_{k,\ell}, \hat{s}_{k,\ell})$ by sampling from $p(x_k, s_k|\hat{s}_{k-1,\ell})$.
3. Compute an estimate of λ_k.
4. Resampling: draw N samples from list $\{(\hat{s}_{k-1,\ell}, \hat{x}_{k,\ell}, \hat{s}_{k,\ell})\}_{\ell=1}^{N}$ by choosing each three-tuple with probability proportional to $p_{Y_k|X_k,S_k,S_{k-1}}(y_k|\hat{x}_{k,\ell}, \hat{s}_{k,\ell}, \hat{s}_{k-1,\ell})$.
5. Drop $\hat{s}_{k-1,\ell}$ and $\hat{x}_{k,\ell}$ of each new three-tuple and obtain the new list $\{\hat{s}_{k,\ell}\}_{\ell=1}^{N}$.

Particle method

Algorithm





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5. Drop $\hat{s}_{k-1,\ell}$ and $\hat{x}_{k,\ell}$ of each new three-tuple and obtain the new list $\{\hat{s}_{k,\ell}\}_{\ell=1}^{N}$.

Observation

Sampling from $p(x_k, s_k | \hat{s}_{k-1,\ell})$

Closed-form expression for $p(x_k, s_k | \hat{s}_{k-1,\ell})$ not required!

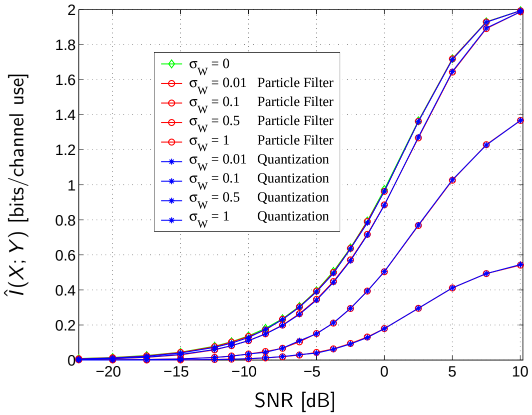
State transitions may be modeled by

- Stochastic finite difference equations
- Stochastic ordinary differential equations
- Stochastic partial differential equations.

Allows detailed physical modeling of communications channel e.g., optical fibers, wave guides, hard drives etc.

Numerical results

Random-walk phase model with i.u.d. 4-PSK input symbols X



Contributions

Modeling

Which physical mechanisms are responsible for (phase) noise?
How can (phase) noise be modeled?

Simple intuitive model for phase noise.

Algorithms

How can phase-estimation algorithms systematically be derived?

As message passing on factor graph of the system a hand.

Performance limits

How well can the (noisy) carrier phase be estimated?
How much does the information rate decrease due to phase noise?

Computation of Cramér-Rao bounds/information rates/capacities.

Capacity of continuous memoryless channel

Definition

Given: memoryless channel with law $p(y|x)$

$$C(X;Y) \triangleq \sup_{p(x)} \int_x \int_y p(x)p(y|x) \log \frac{p(y|x)}{p(y)} dxdy \triangleq \sup_{p(x)} I(X;Y)$$

Discrete input alphabet $\mathcal{X}$

Blahut-Arimoto algorithm.

Continuous input alphabet $\mathcal{X}$

Particle-based approach: $p(x) \approx \{(\hat{x}_1, w_1), (\hat{x}_2, w_2), \ldots, (\hat{x}_N, w_N)\}$.

Method for channels with memory currently in development.

Blahut-Arimoto algorithm

1. START with some $p^{(0)}(x)$.
2. ITERATE

$$\boxed{p^{(k)}(x) = \frac{1}{Z^{(k)}} p^{(k-1)}(x) \exp\left(D\left(p(y|x) \| p^{(k-1)}(y)\right)\right)}$$

$$p^{(k)}(y) \triangleq \int_{x \in \mathcal{X}} p^{(k)}(x) p(y|x) dx.$$

$$Z^{(k)} \triangleq \int_{x \in \mathcal{X}} p^{(k-1)}(x) \exp\left(D\left(p(y|x) \| p^{(k-1)}(y)\right)\right) dx.$$

UNTIL

$$\boxed{\max_{x \in \mathcal{X}} D\left(p(y|x) \| p^{(n)}(y)\right) - I^{(n)} < \varepsilon}$$

$$I^{(n)} \triangleq \int_{x \in \mathcal{X}} p^{(n)}(x) D\left(p(y|x) \| p^{(n)}(y)\right) dx.$$

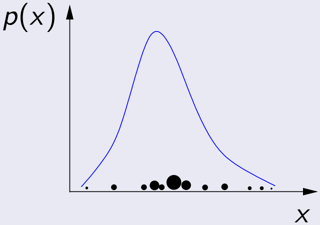
Continuous channels

Blahut-Arimoto algorithm and continuous channels

- Blahut-Arimoto algorithms only practical for discrete channels.
- Continuous channels

$$p(x) = \mathcal{L} \triangleq \{(\hat{x}_1, w_1), (\hat{x}_2, w_2), \ldots, (\hat{x}_N, w_N)\}$$

with $\hat{x}_k \in \mathcal{X}$ and $0 \leq w_k \leq 1$.



Algorithm

Non-convex finite-dimensional optimization problem

$$\mathcal{L}^* \triangleq \underset{\hat{x}, w}{\operatorname{argmax}}\, I(\hat{x}, w)$$

Solved by alternating maximization

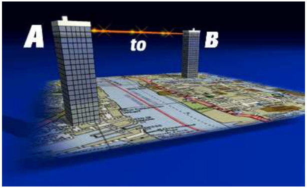
$$w^{(k)} \triangleq \underset{w}{\operatorname{argmax}}\, I\big(\hat{x}^{(k-1)}, w\big) \qquad \text{(Blahut-Arimoto)}$$

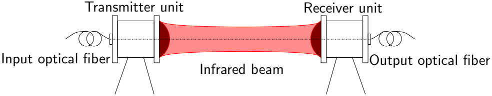
$$\hat{x}^{(k)} \triangleq \underset{\hat{x}}{\operatorname{argmax}}\, I\big(\hat{x}, w^{(k)}\big) \qquad \text{(gradient method)}$$

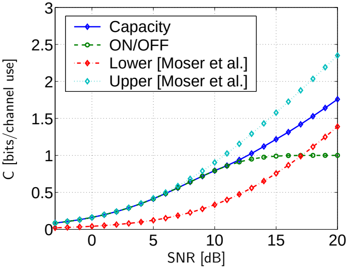
Method for channels with memory currently in development.

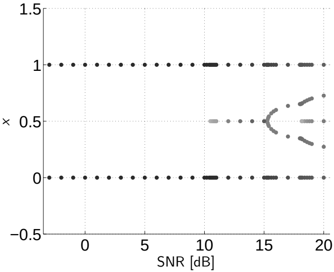
Results: Gaussian channel

$$Y_k = X_k + N_k \text{ with } N_k \overset{\text{i.i.d.}}{\sim} \mathcal{N}_{(0,\sigma_N^2)} \text{ and } \Pr[0 \le X_k \le 1] = 1$$



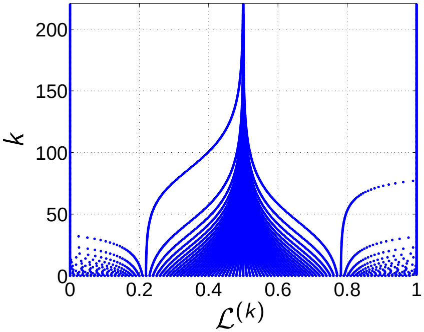






Evolution of the mass points

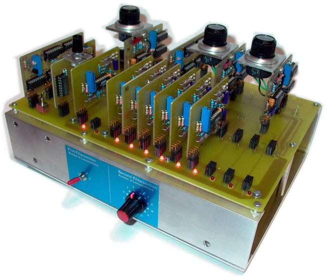
SNR = 13dB

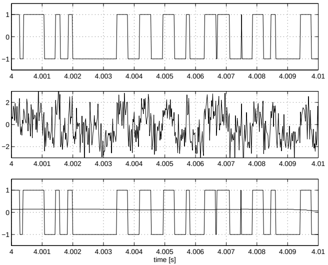


Analog circuit for PN-synchronization

Objective

Analog circuit that locks unto pseudo-random sequences (GPS, UWB, CDMA).





SNR = 0dB

Joint work with M. Frey, N. Gershenfeld, T. Koch, P. Merkli, B. Vigoda.

Analog circuit for PN-synchronization

Objective

Analog circuit that locks unto pseudo-random sequences (GPS, UWB, CDMA).

Method

Discrete-time message-passing algorithm for synchronization to LFSR-sequences converted into continuous-time.

Practical result

Practical circuit built and tested: it works!

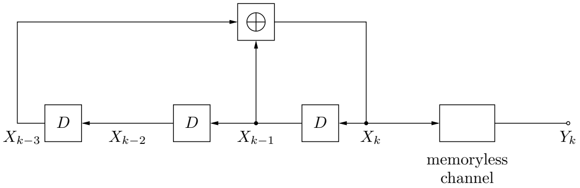
Theoretical result

Connection between entrainment and ideas from estimation theory ("message passing").

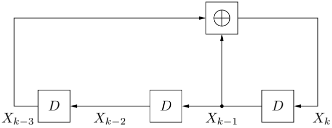
Joint work with M. Frey, N. Gershenfeld, T. Koch, P. Merkli, B. Vigoda.

Discrete-time synchronization task

Based on the noisy observation Y of the LFSR-sequence X, infer the actual state of the source.

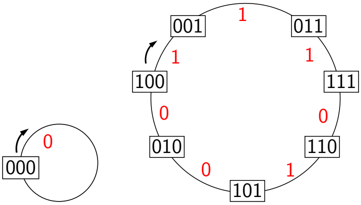


Discrete-time synchronization task

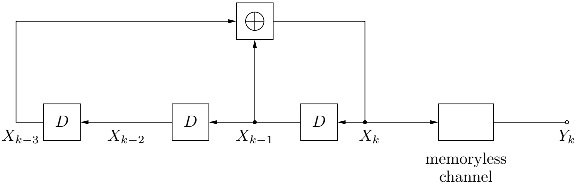


$$X = [\ldots, X_{k-1}, X_k, X_{k+1}, \ldots] \text{ with } X_k = X_{k-1} \oplus X_{k-3}$$

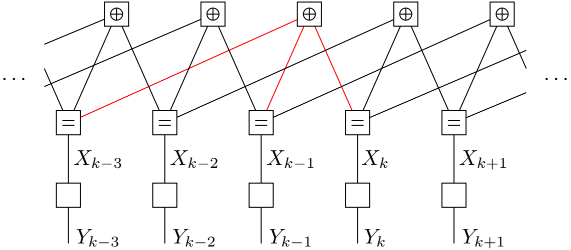
State diagram



Discrete-time synchronization task

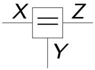
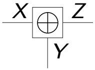


$$X = [\ldots, X_{k-1}, X_k, X_{k+1}, \ldots] \text{ with } X_k = X_{k-1} \oplus X_{k-3}$$

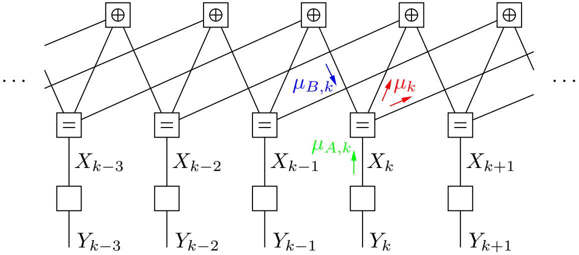


Reminder: SP for EQU and XOR-node

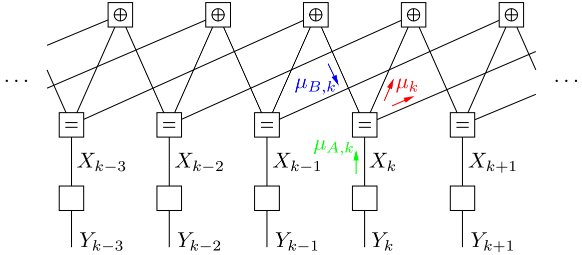
$$L \triangleq \log \frac{\mu(0)}{\mu(1)} \qquad \Delta = \frac{\mu(0)-\mu(1)}{\mu(0)+\mu(1)}$$

 $\delta[x-y]\delta[x-z]$	$$\begin{pmatrix} \mu_Z(0) \\ \mu_Z(1) \end{pmatrix} = \begin{pmatrix} \mu_X(0)\mu_Y(0) \\ \mu_X(0)\mu_X(1) \end{pmatrix}$$ $$L_Z = L_X + L_Y$$ $$\Delta_Z = \frac{\Delta_X+\Delta_Y}{1+\Delta_X\Delta_Y}$$
 $\delta[x \oplus y \oplus z]$	$$\begin{pmatrix} \mu_Z(0) \\ \mu_Z(1) \end{pmatrix} = \begin{pmatrix} \mu_X(0)\mu_Y(0) + \mu_X(1)\mu_Y(1) \\ \mu_X(0)\mu_X(1) + \mu_X(1)\mu_X(0) \end{pmatrix}$$ $$\tanh(L_Z/2) = \tanh(L_X/2) \cdot \tanh(L_Y/2)$$ $$\Delta_Z = \Delta_X\Delta_Y$$

Forward-only message passing on the factor graph

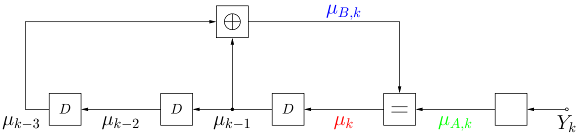


Forward-only message passing on the factor graph



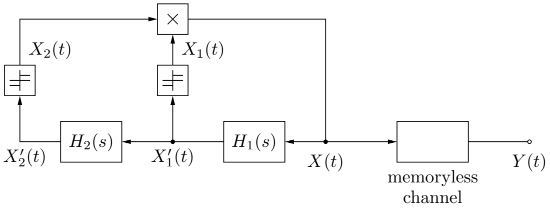
Interpretation:

Filtering of the sequence Y with a soft version of the LFSR.



Reminder: SP for EQU and XOR-node

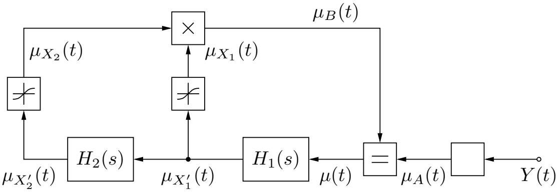
Signal Source



- Delay elements replaced by linear filters.
- Output of the filters $X'_1(t)$ and $X'_2(t) \in \mathbb{R}$.
- Introduction of threshold functions
  $(X_1(t), X_2(t)$ and $X(t) \in \{-1, +1\})$.
- Multiplication corresponds to addition modulo 2.

From Discrete-Time to Continuous-Time

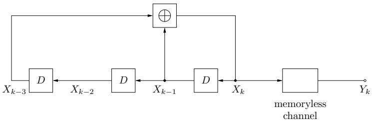
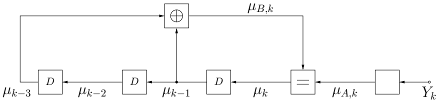
Synchronizing Circuit



A soft version of the signal source.

From Discrete-Time to Continuous-Time (2)

Overview

<table>
<tr><th>Transmitter + Channel</th><th>Receiver</th></tr>
<tr><td>
</td><td>
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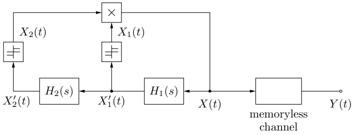
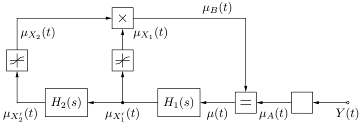
<tr><td>
</td><td>
</td></tr>
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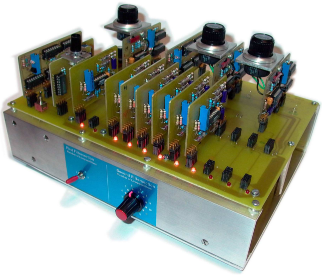
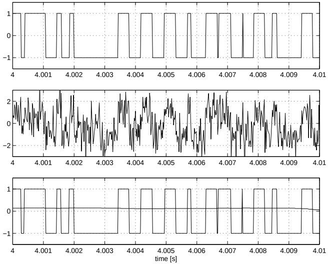
Results





SNR = 0dB

Photograph and measurements by M. Frey and P. Merkli.
Hardware built by T. Schaerer.

Summary

- Framework for deriving inference algorithms:
  - Factor graph = graphical representation of system
  - Algorithm = updating messages on factor graph.
- Message-passing algorithms for computing:
  - information rates
  - channel capacities
  - Cramér-Rao-type bounds.
- Analog circuit for PN-synchronization
  = message-passing algorithm as dynamical system.

Outlook

- Lower bounds on the MSE for discrete variables.
- Extension of the particle-based BA-algorithm to continuous channels with memory/feedback/side information.
- Analog electronic circuits for estimation.
- Novel applications of message-passing methods
    - Information geometry
    - Kernel methods.

On various applications of message passing on factor graphs

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December 13

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Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Noise Sources

Shot noise

Fluctuations of electrical currents due to discreteness of charges:

$$I(t) = \sum_k q\delta(t - t_k), \quad t_k \sim \text{Poisson process}$$

Zero-mean white noise $\tilde{I}(t) \triangleq I(t) - \mathrm{E}[I(t)]$ with spectral density:

$$S_{\text{shot}}(f) = 2q\mathrm{E}[I(t)].$$

Thermal noise

Fluctuations of open-circuit voltage and closed-circuit current of conductor. Is nothing but shot noise! [Sarpeshkar et al., 1993]

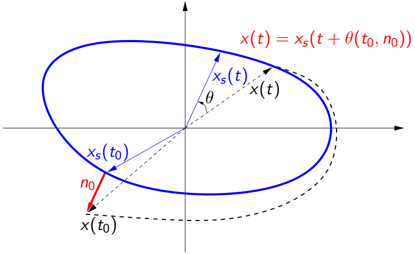
Flicker noise (or "1/f noise")

Fluctuations of inter-event time $\tau_k \triangleq t_k - t_{k-1}$ modeled by AR(1) model. [Kaulakys, 2004]

Phase Noise in Free-Running Clocks

Perturbed autonomous system

$$\frac{dx}{dt} = f(x) + N(t), \quad x \in \mathbb{R}^n, \quad N(t) \text{ is “noise”}.$$



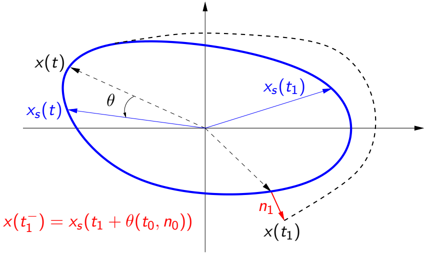
Phase offset due to “small” perturbation at $t = t_0$

$$\theta(t_0, n_0) \approx \gamma(x_s(t_0)) \cdot n_0.$$

Phase Noise in Free-Running Clocks

Perturbed autonomous system

$$\frac{dx}{dt} = f(x) + N(t), \quad x \in \mathbb{R}^n, \quad N(t) \text{ is "noise"}.$$



Phase offset due to "small" perturbation at $t = t_1 \gg t_0$

$$\theta(t_0, n_0, t_1, n_1) \approx \gamma(x_s(t_0)) \cdot n_0 + \gamma(x_s(t_1 + \theta(x_s(0), n_0)) \cdot n_1.$$

Phase Noise in Free-Running Oscillator

Continuous-time phase-noise model

$$\Theta(t) = \left[\int_{-\infty}^{t} \gamma(t' + \Theta(t')) \cdot N(t')dt'\right] \bmod 2\pi.$$

Discrete-time phase-noise model

$$\Theta_k = \big(\Theta_{k-1} + N_k\big) \bmod 2\pi, \quad \Theta_k \triangleq \Theta(kT_s)$$

with

$$N_k \triangleq \int_{(k-1)T_s}^{kT_s} \gamma(t' + \Theta(t'))N(t')dt'.$$

Discrete-time phase-noise model: white-noise sources

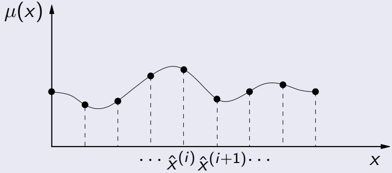
$$\Theta_k = \big(\Theta_{k-1} + N_k\big) \bmod 2\pi, \quad N_k \sim \mathcal{N}_{0,\sigma_N^2}.$$

Numerical integration

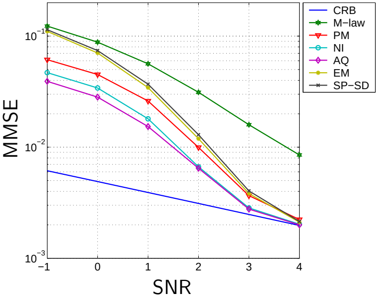
Integral-product rule evaluated by numerical integration

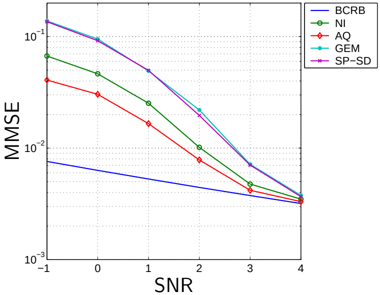
$$\mu_{f\to Y}(y) \propto \sum_{i_1,\ldots,i_N} f\left(y, \hat{x}_1^{(i_1)}, \ldots, \hat{x}_N^{(i_N)}\right) \cdot \mu_{X_1\to f}\left(\hat{x}_1^{(i_1)}\right) \cdots \mu_{X_N\to f}\left(\hat{x}_N^{(i_N)}\right),$$

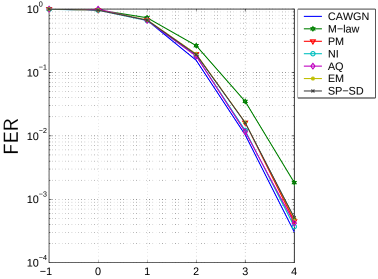
where $\hat{x}_k^{(i_k)}$ is the i_k-th quantization level of X_k.

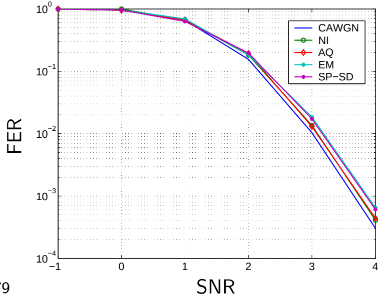


Phase estimation: results

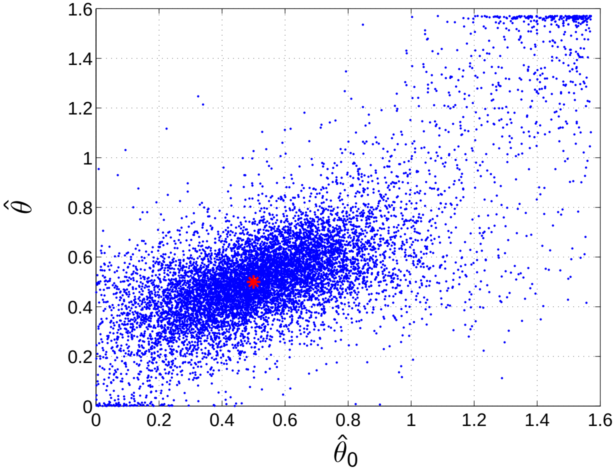








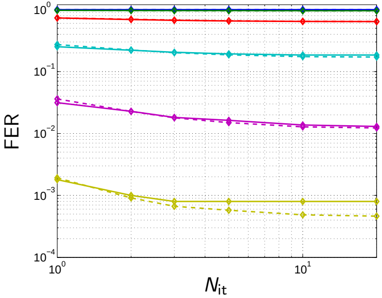
EM: initial estimate vs. final estimate



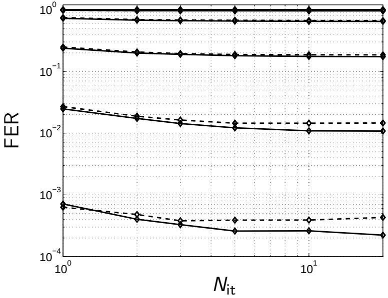
Convergence

Random-walk phase model

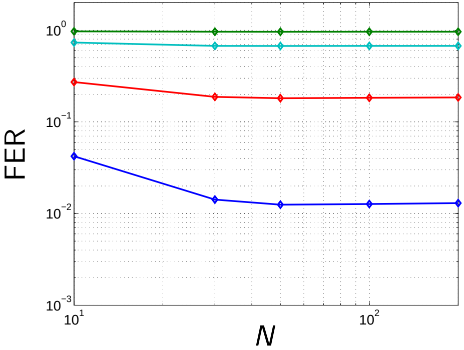
EM



Quantization



FER as a function of number of quantization levels



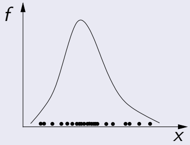
(SNR = 0dB, 1dB, 2dB, and 3dB)

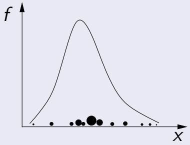
Particle methods

Integral-product rule evaluated by particle methods

$$\mu_{f\to Y}(y) \;\propto\; \sum_{i_1,\ldots,i_N} f(y, \hat{x}_1^{(i_1)}, \ldots, \hat{x}_N^{(i_N)}) \cdot w_1^{(i_1)} \cdots w_N^{(i_N)},$$

where $\hat{x}_k^{(i_k)}$ is the i_k-th particle of the particle list that represents $\mu_{X_k\to f}$, and $w_k^{(i_k)}$ is the weight of that particle.





Gibbs sampling

Algorithm

1. Choose an initial value $(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_N)$.
2. Choose an index k.
3. Draw a sample $\hat{x}_k$ from

$$f(x_k) \triangleq \frac{f(\hat{x}_1, \ldots, \hat{x}_{k-1}, x_k, \hat{x}_{k+1}, \ldots, \hat{x}_N)}{\sum_{x_k} f(\hat{x}_1, \ldots, \hat{x}_{k-1}, x_k, \hat{x}_{k+1}, \ldots, \hat{x}_N)}.$$

4. Iterate 2–3 a “large” number of times.

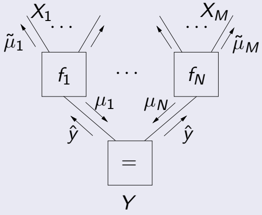
Gibbs sampling

Message passing view

1. Select variable (EQU constraint node) Y in FG of f.
2. EQU node Y generates message $\hat{y}$ by sampling from:

$$f(y) \triangleq \frac{\mu_1(y) \ldots \cdot \mu_N(y)}{\sum_y \mu_1(y) \ldots \cdot \mu_N(y)},$$

   and broadcasts $\hat{y}$ to its neighboring nodes f_k $(k = 1, \ldots, M)$.
3. Nodes f_k update messages $\tilde{\mu}$ by applying sum-product rule with as incoming messages the samples $\hat{y}$ and $\hat{x}_\ell$ $(\ell = 1, \ldots, M)$.



Importance sampling

Suppose we wish to compute:

$$\mathrm{E}_f[g] \triangleq \int_x f(x)g(x), \tag{1}$$

but naive computation is intractable.
Generate a list samples $\left\{\hat{x}^{(i)}\right\}_{i=1}^{N}$ from f and evaluate (1) as

$$\mathrm{E}_f[g] \triangleq \frac{1}{N}\sum_{i=1}^{N} g\left(\hat{x}^{(i)}\right). \tag{2}$$

Suppose that sampling from f is "hard", hence (2) is infeasible.
Draw samples $\left\{\hat{x}^{(i)}\right\}_{i=1}^{N}$ from a different function h
with $\mathrm{supp}(f) \subseteq \mathrm{supp}(h)$, compute (1) as

$$\mathrm{E}_f[g] \triangleq \frac{1}{N}\sum_{i=1}^{N} w^{(i)} g\left(\hat{x}^{(i)}\right) \text{ with } w^{(i)} \triangleq \frac{f\left(\hat{x}^{(i)}\right)}{h\left(\hat{x}^{(i)}\right)}. \tag{3}$$

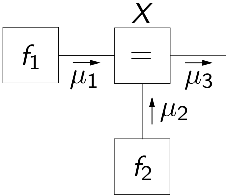
Importance sampling (2)

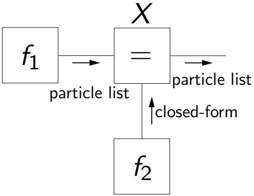
Suppose

$$f(x) \triangleq f_1(x)f_2(x).$$

Draw samples $\left\{\hat{x}^{(i)}\right\}_{i=1}^{N}$ from f_1 and weight those samples by the function f_2:

$$w^{(i)} \triangleq f_2\left(\hat{x}^{(i)}\right).$$



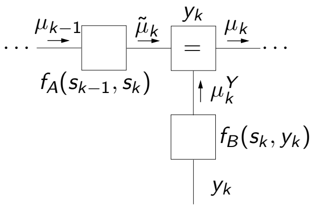


Particle filtering

Particle filtering (or "sequential Monte-Carlo integration")
= forward-only message passing in a state-space model:

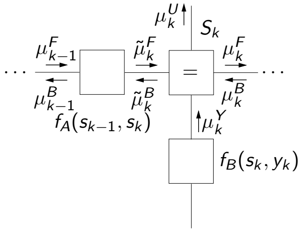
$$f(s_0, s_2, \ldots, s_N, y_1, y_2, \ldots, y_N) \triangleq f_A(s_0) \prod_{k=1}^{N} f_A(s_{k-1}, s_k) f_B(s_k, y_k),$$

where messages are represented by lists of samples.

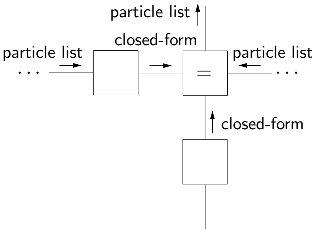


$\tilde{\mu}_k$ is obtained from μ_{k-1} by weighted or unweighted sampling.
μ_k is generated from $\tilde{\mu}_k$ by importance sampling.

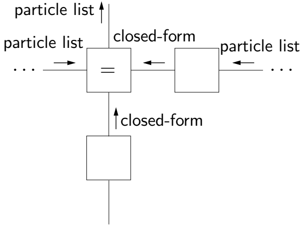
Smoothing



μ_k^U
S_k
μ_{k-1}^F
$\tilde{\mu}_k^F$
μ_k^F
$\cdots$
$=$
$\cdots$
μ_{k-1}^B
$\tilde{\mu}_k^B$
μ_k^B
$f_A(s_{k-1}, s_k)$
μ_k^Y
$f_B(s_k, y_k)$



particle list
closed-form
particle list
particle list
$\cdots$
$=$
$\cdots$
closed-form



particle list
particle list
closed-form
particle list
$\cdots$
$=$
$\cdots$
closed-form

MCMC

Algorithm

1. Choose an initial value $\hat{x}$.
2. Sample $\hat{y}$ from $q(y|\hat{x})$.
3. Set
$$\hat{x} \triangleq \hat{y} \quad \text{with probability } p$$
where
$$p \triangleq \min\left\{\frac{f(\hat{y})}{f(\hat{x})}, 1\right\}$$
4. Iterate 2–3 a sufficient number of times.

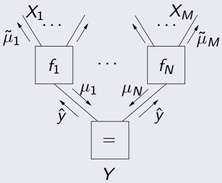
MCMC

Message passing view

1. Select variable (equality constraint node) Y in FG of f.
2. Edge Y generates the message $\hat{y}^{\text{new}}$ by sampling from $q(y|\hat{y})$.
3. Set $\hat{y} \triangleq \hat{y}^{\text{new}}$ with probability p where

$$p \triangleq \min\left\{\frac{f(\hat{y}^{\text{new}})}{f(\hat{y})}, 1\right\} \text{ with } f(y) \triangleq \frac{\mu_1(y)\ldots\mu_N(y)}{\sum_y \mu_1(y)\ldots\mu_N(y)}.$$

   The message $\hat{y}$ is broadcast to the neighboring nodes f_k.
4. Nodes f_k update outgoing messages $\tilde{\mu}$ by applying the SP rule with as incoming messages the samples $\hat{y}$ and $\hat{x}_\ell$.



Simulated Annealing

Objective

- to sample from a multivariate function $f(x_1, \ldots, x_N)$,
- to find the mode of the function f.

Algorithm

1. Choose an initial value $(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_N)$.
2. Choose an initial value α (e.g., $\alpha = 0.1$).
3. Sample a new value $\hat{y}$ from $q(y|\hat{x})$.
4. Set $\hat{x} \triangleq \hat{y}$ with probability p, where
$$p \triangleq \min\left\{\left(\frac{f(\hat{y})}{f(\hat{x})}\right)^{\alpha}, 1\right\}$$
5. Iterate 3–4 a “large” number of times.
6. Increase α according to some schedule.
7. Iterate 5–6 until convergence or until available time is over.

Single value approximation

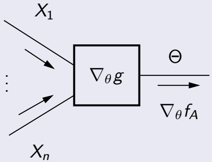
Integral-product rule evaluated by means of hard decision

$$\mu_{f\to Y}(y) \quad \propto \quad f(y, \hat{x}_1, \ldots, \hat{x}_N),$$

where $\hat{x}_k$ is a hard estimate of X_k, representing the message $\mu_{X_k\to f}$.

Gradient descent / sum-product

$$\nabla_\theta f_A(\theta) \propto \sum_{x_1,\ldots,x_n} \nabla_\theta g(x_1, \ldots, x_n, \theta) \cdot \prod_{\ell=1}^{n} \mu_{X_\ell \to g}(x_\ell).$$



Expectation Maximization: General problem

$$\boxed{\theta_{\max} \triangleq \operatorname*{argmax}_{\theta} f(\theta)}$$

θ takes values in $\mathbb{R}$ or $\mathbb{R}^n$

$$\boxed{f(\theta) \triangleq \int_x f(x,\theta)dx}$$

$\int_x g(x)dx$ stands for summation or integration.

In principle

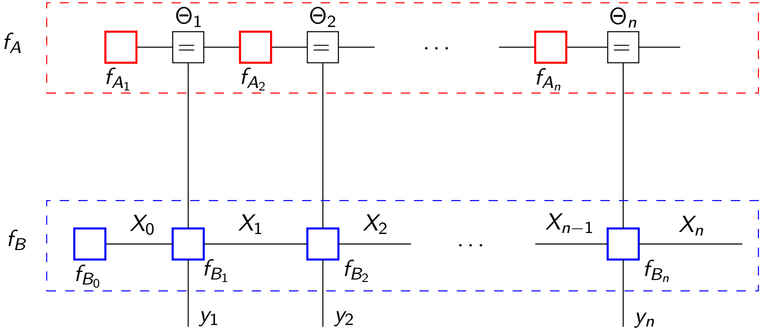
1. Determine $f(\theta)$ by sum-product message passing
2. $\theta_{\max} \triangleq \operatorname{argmax}_{\theta} f(\theta)$ by max-product message-passing

Often infeasible, since

- Sum-product rule may lead to intractable integrals
- Maximization step may be infeasible.

Parameter estimation in state-space model

$$f(x,\theta) = f_A(\theta_1)\prod_{k=1}^{n-1} f_A(\theta_k,\theta_{k+1}) \cdot f_B(x_0)\prod_{k=1}^{n} f_B(x_{k-1},x_k,\theta_k,y_k)$$



Expectation Maximization

1. Make initial guess $\theta^{(0)}$
2. Expectation step

$$\boxed{f^{(\ell)}(\theta) \triangleq \int_x f(x, \hat{\theta}^{(\ell)}) \log f(x, \theta) dx}$$

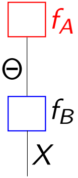
3. Maximization step

$$\boxed{\theta^{(\ell+1)} \triangleq \operatorname*{argmax}_{\theta} f^{(\ell)}(\theta)}$$

4. Repeat 2–3 until convergence.

EM as message passing

$$f(x,\theta) \triangleq f_A(\theta) f_B(x,\theta)$$



EM as message passing (2)

Expectation step

$$\boxed{f^{(\ell)}(\theta) \triangleq \int_x f(x, \hat{\theta}^{(\ell)}) \log f(x, \theta) dx}$$

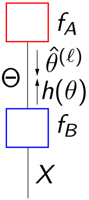
Maximization step

$$\boxed{\theta^{(\ell+1)} \triangleq \underset{\theta}{\operatorname{argmax}}\, f^{(\ell)}(\theta)}$$

$$\begin{aligned}
\hat{\theta}^{(\ell+1)} &= \underset{\theta}{\operatorname{argmax}} \int_x f(x, \hat{\theta}^{(\ell)}) \log f(x, \theta) dx \\
&= \underset{\theta}{\operatorname{argmax}} \int_x f_A\big(\hat{\theta}^{(\ell)}\big) f_B\big(x, \hat{\theta}^{(\ell)}\big) \log \big(f_A(\theta) f_B(x, \theta)\big) dx \\
&= \underset{\theta}{\operatorname{argmax}} \int_x f_B\big(x, \hat{\theta}^{(\ell)}\big) \big(\log f_A(\theta) + \log f_B(x, \theta)\big) dx \\
&= \underset{\theta}{\operatorname{argmax}} \left(\log f_A(\theta) + \frac{\int_x f_B\big(x, \hat{\theta}^{(\ell)}\big) \log f_B(x, \theta) dx}{\int_x f_B\big(x, \hat{\theta}^{(\ell)}\big) dx} \right) \\
&= \operatorname{argmax} \big(\log f_A(\theta) + h(\theta)\big)
\end{aligned}$$

EM as message passing (3)

$$f(x,\theta) \triangleq f_A(\theta) f_B(x,\theta)$$



Upwards message $h(\theta)$

$$\begin{aligned} h(\theta) &= \frac{\int_x f_B(x,\hat{\theta}^{(\ell)}) \log f_B(x,\theta) dx}{\int_x f_B(x,\hat{\theta}^{(\ell)}) dx} \\ &= \mathrm{E}_{p_B}[\log f_B(x,\theta)] \end{aligned}$$

$$p_B(x|\hat{\theta}^{(\ell)}) \triangleq \frac{f_B(x,\hat{\theta}^{(\ell)})}{\int_x f_B(x,\hat{\theta}^{(\ell)}) dx}$$

Downwards message $\hat{\theta}^{(\ell+1)}$

$$\hat{\theta}^{(\ell+1)} = \underset{\theta}{\operatorname{argmax}} \left(\log f_A(\theta) + h(\theta) \right)$$

EM as message passing (4)

Remarks

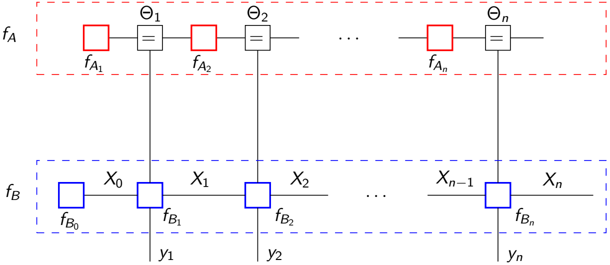
- If $f_A(\theta)$ is constant, then normalization may be omitted.
- Message $h(\theta)$ is not sum-product message!
- f_A and f_B often have a "nice" structure.

EM as message passing (5)

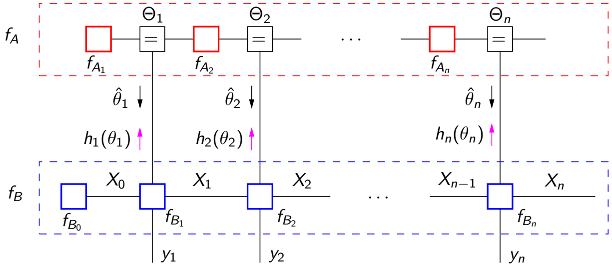
Trellis and state space models

$$f_A(\theta) \triangleq f_{A_1}(\theta_1) f_{A_2}(\theta_1, \theta_2) \ldots f_{A_n}(\theta_{n-1}, \theta_n)$$

$$f_B(x, \theta) \triangleq f_{B_0}(x_0) f_{B_1}(x_0, x_1, y_1, \theta_1) f_{B_2}(x_1, x_2, y_2, \theta_2) \ldots f_{B_n}(x_{n-1}, x_n, y_n, \theta_n)$$



EM as message passing (6)



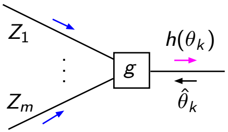
$$h(\theta) = \sum_{\ell=1}^{n} h_\ell(\theta_\ell) = \sum_{\ell=1}^{n} \int_{x_{\ell-1}} \int_{x_\ell} p_B(x_{\ell-1}, x_\ell, |y, \hat\theta) \log f_{B_\ell}(x_{\ell-1}, x_\ell, y, \theta_\ell) dx_{\ell-1} dx_\ell$$

$$p_B(x_{\ell-1}, x_\ell, |y, \hat\theta) = \frac{f_{B_\ell}(x_{\ell-1}, x_k, y, \theta_\ell) \mu_{X_\ell \to f_{B_\ell}}(x_\ell) \mu_{X_{\ell-1} \to f_{B_\ell}}(x_{\ell-1})}{\int_{x_{\ell-1}} \int_{x_\ell} f_{B_\ell}(x_{\ell-1}, x_\ell, y, \theta_k) \mu_{X_\ell \to f_{B_\ell}}(x_\ell) \mu_{X_{\ell-1} \to f_{B_\ell}}(x_{\ell-1}) dx_{\ell-1} dx_\ell}$$

$$(\hat\theta_1, \hat\theta_2, \ldots, \hat\theta_n)^T = \underset{\theta_1, \theta_2, \ldots, \theta_n}{\operatorname{argmax}} \left[\log f_{A_1}(\theta_1) + \sum_{\ell=2}^{n} \log f_{A_\ell}(\theta_{\ell-1}, \theta_\ell) + \sum_{\ell=1}^{n} h_\ell(\theta_\ell) \right]$$

EM as message passing (7)

h-messages



$$\begin{aligned}
h(\theta_k) &= \gamma^{-1} \int_z g(z_1, \ldots, z_m, \hat{\theta}_k)\mu(z_1) \ldots \mu(z_m) \log g(z_1, \ldots, z_m, \theta_k) dz \\
&= \int_z p(z_1, \ldots, z_m | \hat{\theta}_k) \log g(z_1, \ldots, z_m, \theta_k) dz \\
&= \mathrm{E}_{p(z_1, \ldots, z_m | \hat{\theta}_k)}[\log g(z_1, \ldots, z_m, \theta_k)]
\end{aligned}$$

$$\begin{aligned}
p(z_1, \ldots, z_m | \hat{\theta}_k) &= \gamma^{-1} g(z_1, \ldots, z_m, \hat{\theta}_k)\mu(z_1) \ldots \mu(z_m) \\
\gamma &= \int_z g(z_1, \ldots, z_m, \hat{\theta}_k)\mu(z_1) \ldots \mu(z_m) dz
\end{aligned}$$

$\mu(z_k)$ are sum-product messages

Expectation Maximization: Properties

Theorem (Main property)

$$f(\hat{\theta}^{(k+1)}) \geq f(\hat{\theta}^{(k)}).$$

Corollary

The global maximum θ^{max} of $f(\theta)$ is a fixed point of EM.

Theorem

The fixed points of EM are stationary points of $f(\theta)$.

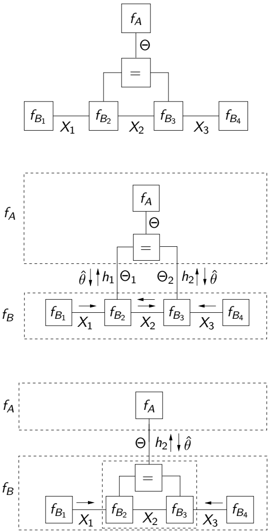
Theorem

A stationary point $\hat{\theta}^{stat}$ of f is a fixed point of EM, if $\bar{f}(\theta, \hat{\theta}^{stat})$ with

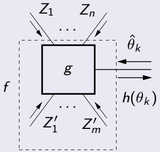
$$\bar{f}(\theta, \theta') \triangleq \sum_x f(x, \theta') \log f(x, \theta),$$

is concave in θ.

EM and compound nodes



Hybrid EM



$$h(\theta_k) = \sum_{z_1} \dots \sum_{z_n} p(z_1, \dots, z_n | \hat{\theta}_k) \log f(z_1, \dots, z_n, \theta_k),$$

$$= \gamma^{-1} \sum_{z_1} \dots \sum_{z_n} f(z_1, \dots, z_n, \hat{\theta}_k)\, \mu(z_1) \cdots \mu(z_n)$$

$$\cdot \log f(z_1, \dots, z_n, \theta_k),$$

with

$$\gamma \triangleq \sum_{z_1} \dots \sum_{z_n} f(z_1, \dots, z_n, \hat{\theta}_k)\, \mu(z_1) \cdots \mu(z_n),$$

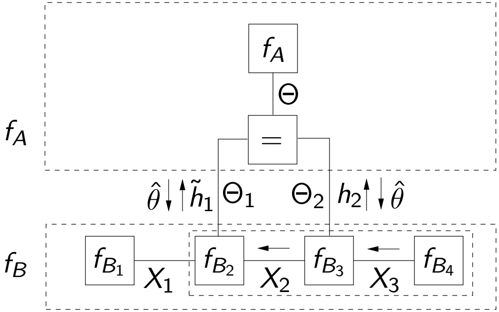
and

$$f(z_1, \dots, z_n, \theta_k) \propto \sum_{z'_1} \dots \sum_{z'_m} g(z_1, \dots, z_n, z'_1, \dots, z'_m, \theta_k)$$

$$\cdot \mu(z'_1) \cdots \mu(z'_m),$$

where $\mu(z_1), \dots, \mu(z_n), \mu(z'_1), \dots, \mu(z'_m)$ are standard sum-product messages.

Example


f_A
Θ
$=$
f_A
$\hat{\theta}$
$\tilde{h}_1$
Θ_1
Θ_2
h_2
$\hat{\theta}$
f_B
f_{B_1}
X_1
f_{B_2}
X_2
f_{B_3}
X_3
f_{B_4}

(Hybrid) EM: properties

Theorem (Cycle-free $f_B(x,\theta)$)

Assume that a factor graph of a global function $f(x,\theta) \triangleq f_A(\theta)f_B(x,\theta)$ is available whose subgraph $f_B(x,\theta)$ is cycle-free. The fixed points of a hybrid EM algorithm applied on that factor graph are stationary points of the marginal $f(\theta)$.

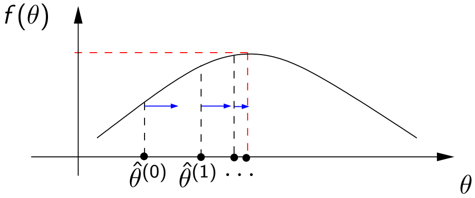
Theorem (Cyclic $f_B(x,\theta)$)

Assume that a factor graph of a global function $f(x,\theta) \triangleq f_A(\theta)f_B(x,\theta)$ is available (whose subgraph $f_B(x,\theta)$ may be cycle-free or cyclic). The fixed points of a (hybrid) EM algorithm applied on that factor graph are stationary points of the function $\hat{f}(\theta)$, defined as:

$$\log \hat{f}(\theta) \triangleq \log f_A(\theta) + \int_{-\infty}^{\theta} E_{b(x|\tilde{\theta})}\left[\nabla_\theta \log f_B(x,\tilde{\theta})\right] d\tilde{\theta},$$

where the beliefs $b(\cdot|\theta)$ are computed by means of the sum-product messages available at convergence of the sum-product algorithm.

Steepest descent



Tries to solve

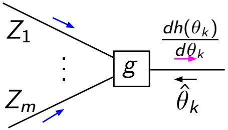
$$\hat{\theta} = \underset{\theta}{\operatorname{argmax}}\, f(\theta)$$

as follows

1. Choose some initial guess $\hat{\theta}^{(0)}$
2. ITERATE
$$\hat{\theta}^{(\ell+1)} = \hat{\theta}^{(\ell)} + \lambda \left.\frac{df(\theta)}{d\theta}\right|_{\theta=\hat{\theta}^{(l)}}$$
3. UNTIL convergence or available time is over

λ is a real positive number referred to as "step size" or "learning rate"

Gradient EM



$$\frac{dh(\theta_k)}{d\theta_k} = \gamma^{-1} \sum_z g(z_1, \ldots, z_m, \hat{\theta}_k)\mu(z_1)\ldots\mu(z_m)\frac{d \log g(z_1, \ldots, z_m, \theta_k)}{d\theta_k} dz,$$

$$= \sum_z p(z_1, \ldots, z_m|\hat{\theta}_k)\frac{d \log g(z_1, \ldots, z_m, \theta_k)}{d\theta_k},$$

$$= \mathrm{E}_{p(z_1,\ldots,z_m|\hat{\theta}_k)}\left[\frac{d \log g(z_1, \ldots, z_m, \theta_k)}{d\theta_k}\right].$$

Gradient EM: Properties

Theorem (Cycle-free $f_B(x,\theta)$)

Assume that a factor graph of a global function $f(x,\theta) \triangleq f_A(\theta) f_B(x,\theta)$ is available whose subgraph $f_B(x,\theta)$ is cycle-free. The fixed points of gradient EM applied on the graph of $f(x,\theta)$ are the stationary points of $f(\theta)$.

Theorem (Cyclic $f_B(x,\theta)$)

Assume that a factor graph of a global function $f(x,\theta) \triangleq f_A(\theta) f_B(x,\theta)$ is available (whose subgraph $f_B(x,\theta)$ may be cycle-free or cyclic). The fixed points of a gradient EM algorithm applied on that factor graph are the stationary points of the function $\hat{f}(\theta)$, defined as:

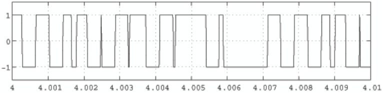
$$\log \hat{f}(\theta) \triangleq \log f_A(\theta) + \int_{-\infty}^{\theta} E_{b(x|\tilde{\theta})}\left[\nabla_\theta \log f_B(x,\tilde{\theta})\right] d\tilde{\theta},$$

where the beliefs $b(\cdot|\theta)$ are computed by means of the sum-product messages available at convergence of the sum-product algorithm.

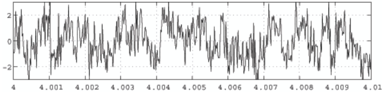
Problem statement

- Pseudo-noise signal X is transmitted over noisy channel, resulting in the noisy signal Y.
- The analog circuit estimates the signal X from the noisy signal Y.

Pseudo-noise signal X



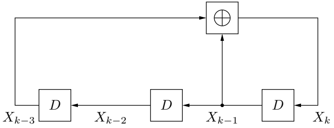
Received noisy signal Y



Applications

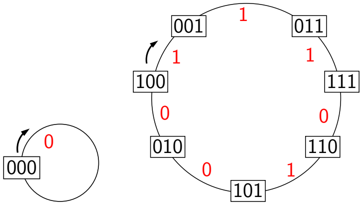
- Spread spectrum communication systems (CDMA, UWB)
- Positioning systems (GPS)

Pseudo-random sequence X generated by LFSR

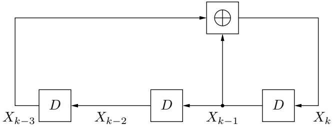


$$X = [\ldots, X_{k-1}, X_k, X_{k+1}, \ldots] \text{ with } X_k = X_{k-1} \oplus X_{k-3}$$

State diagram

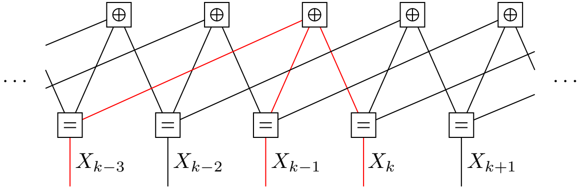


Pseudo-random sequence X generated by LFSR



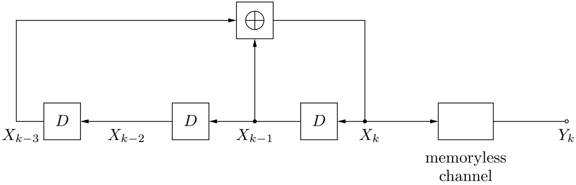
$$X = [\ldots, X_{k-1}, X_k, X_{k+1}, \ldots] \text{ with } X_k = X_{k-1} \oplus X_{k-3}$$

Representation as factor graph.



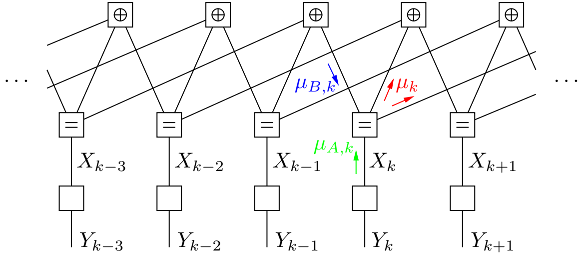
Synchronization task

Based on the noisy observation Y of the sequence X, estimate the actual state of the source.

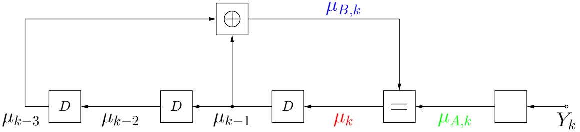


Approach:
Use the factor graph to define a message-passing algorithm.

Forward-only message passing on the factor graph

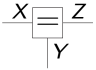
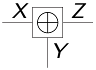


Interpretation:
Filtering of the sequence Y with a soft version of the LFSR.



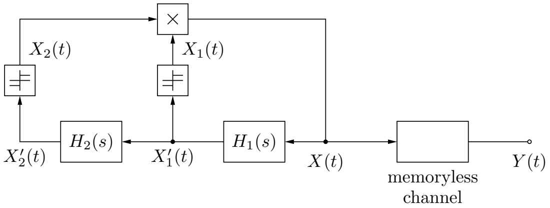
Reminder: SP for EQU and XOR-node

$$L \triangleq \log \frac{\mu(0)}{\mu(1)} \qquad \Delta = \frac{\mu(0)-\mu(1)}{\mu(0)+\mu(1)}$$

 $\delta[x-y]\delta[x-z]$	$\begin{pmatrix} \mu_Z(0) \\ \mu_Z(1) \end{pmatrix} = \begin{pmatrix} \mu_X(0)\mu_Y(0) \\ \mu_X(0)\mu_X(1) \end{pmatrix}$ $L_Z = L_X + L_Y$ $\Delta_Z = \frac{\Delta_X+\Delta_Y}{1+\Delta_X\Delta_Y}$
 $\delta[x \oplus y \oplus z]$	$\begin{pmatrix} \mu_Z(0) \\ \mu_Z(1) \end{pmatrix} = \begin{pmatrix} \mu_X(0)\mu_Y(0) + \mu_X(1)\mu_Y(1) \\ \mu_X(0)\mu_X(1) + \mu_X(1)\mu_X(0) \end{pmatrix}$ $\tanh(L_Z/2) = \tanh(L_X/2) \cdot \tanh(L_Y/2)$ $\Delta_Z = \Delta_X\Delta_Y$

Reminder: SP for EQU and XOR-node

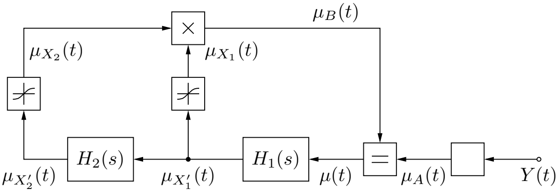
Signal Source



- Delay elements replaced by linear filters.
- Output of the filters $X_1'(t)$ and $X_2'(t) \in \mathbb{R}$.
- Introduction of threshold functions ($X_1(t), X_2(t)$ and $X(t) \in \{-1, +1\}$).
- Multiplication corresponds to addition modulo 2.

From Discrete-Time to Continuous-Time (2)

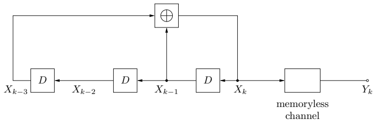
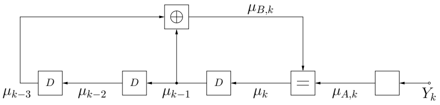
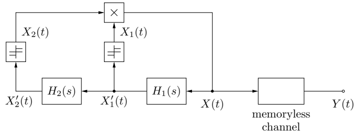
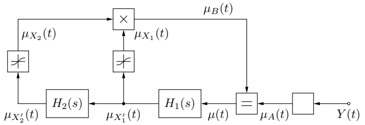
Synchronizing Circuit



A soft version of the signal source.

From Discrete-Time to Continuous-Time (3)

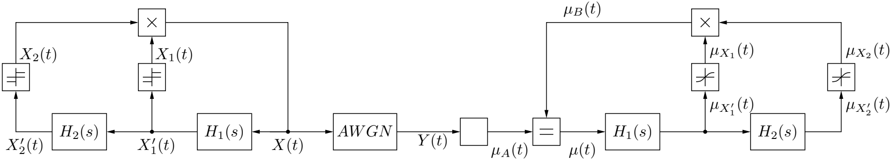
Overview

Transmitter + Channel	Receiver
	
	

Demonstration System

Signals in the receiver are pseudo probability functions of the corresponding signals in the source

- Discrete variables represented as “pseudo-means” (Δ-representation)
  e.g., $\tilde{\mathrm{E}}[X(t)] = \mu_X(t) = \Pr[X(t) = +1] - \Pr[X(t) = -1]$
- Continuous variables ($X'_{1,2}$)
  The pdf for $X'_{1,2}(t)$ is assumed to be Gaussian $\mathcal{N}(\mu_{X'_{1,2}}(t), \sigma^2_{1,2})$.
  - Means $\mu_{X'_{1,2}}$ are computed
  - Variances $\sigma^2_{1,2}$ are fixed and set manually



Demonstration System (2)

Filters

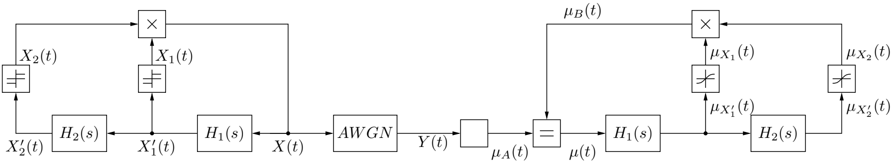
- The means μ_X and $\mu_{X_1'}$ are filtered by $H_1(s)$ and $H_2(s)$. Indeed, let $y(t) = [h \star x](t)$, then $\mathrm{E}[y(t)] = [h \star \mathrm{E}[x]](t)$.
- Remark: for computing the mean, the variance is not needed!

Soft-Thresholds:

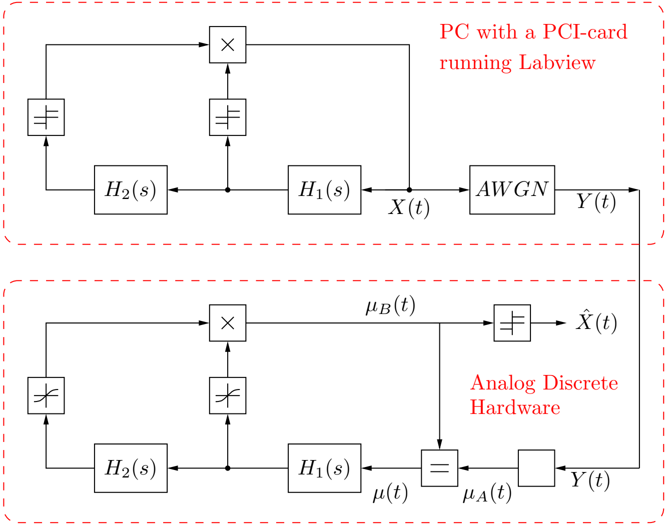
$$\Pr[X_{1,2}(t) = +1] = \Pr[X'_{1,2}(t) \geq 0]$$
$$\Pr[X_{1,2}(t) = -1] = \Pr[X'_{1,2}(t) < 0]$$

$$\mu_{X_{1,2}}(t) = \Pr[X_{1,2}(t) = +1] - \Pr[X_{1,2}(t) = -1] = \operatorname{erf}\left(\frac{\mu_{X'_{1,2}}(t)}{\sqrt{2}\sigma_{1,2}}\right)$$

$$\mu_{X_{1,2}}(t) \approx \tanh\left(C\mu_{X'_{1,2}}(t)\right)$$



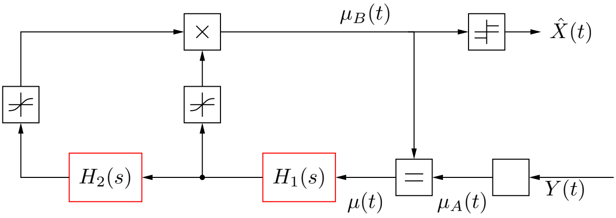
Demonstration System (3)



Demonstration System (4)

The Filters

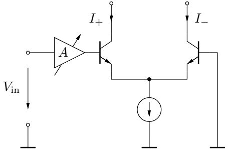
- $H_1(s)$: Butterworth Lowpass Filter, 5^{th} order, $f_c = 1.6$ kHz
- $H_2(s)$: $4 \times H_1(s)$ in series (or $6 \times H_1(s)$ in series)

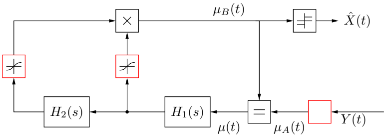


Demonstration System (5)

The Soft-Threshold Function, AWGN-Channel Estimation

Differential pair with the gain A as an adjustable parameter



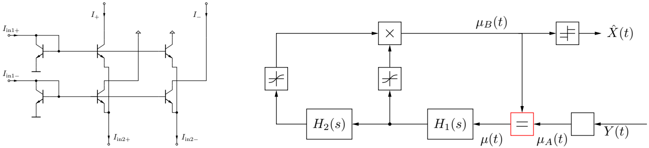


$$\log \frac{\mu_A(0)(t)}{\mu_A(1)(t)} = \log \frac{e^{-\frac{(Y(t)-1)^2}{2\sigma^2}}}{e^{-\frac{(Y(t)+1)^2}{2\sigma^2}}} = \frac{2Y(t)}{\sigma^2}$$

Demonstration System (6)

The Equality Constraint Gate

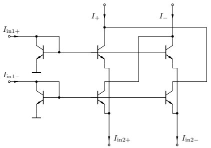
Forward-only EQU-Softgate

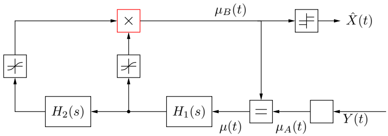


Demonstration System (7)

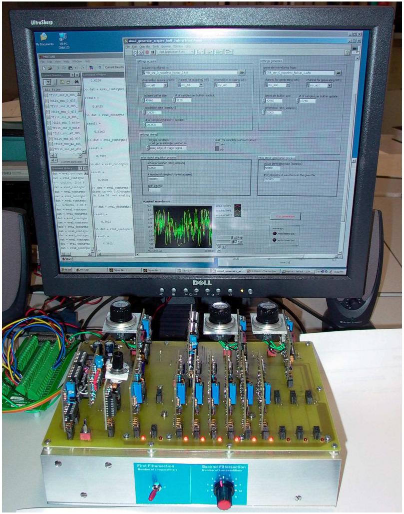
The Soft-XOR Gate

Forward-only XOR-Softgate corresponds to a "Gilbert multiplier"





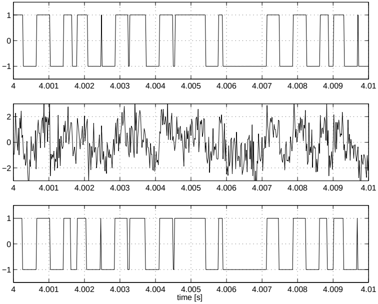
Demonstration System (8)



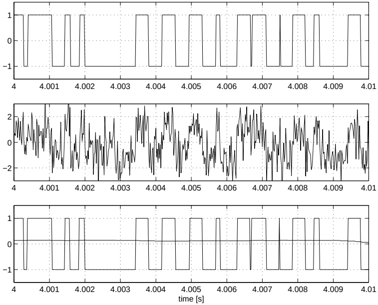
Example of a sequence at SNR = 0 dB

Sampling rate: 50 kHz, 500 samples shown

$H_2(s)$: $4 \times H_1(s)$
sequence-length: 361 samples

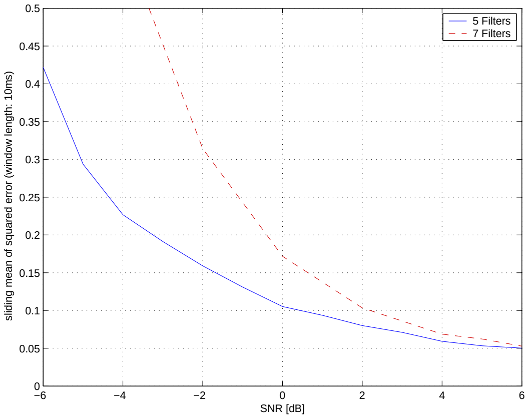


$H_2(s)$: $6 \times H_1(s)$
1'707 samples



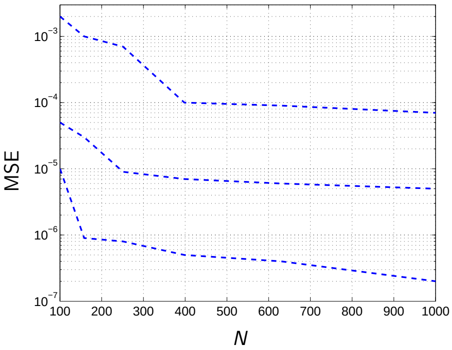
Measurement results

MSE vs. SNR



Results

Results for σ_W^2 ($\sigma_U^2 = 0.1$; σ_W^2=0.001,0.01,0.1)

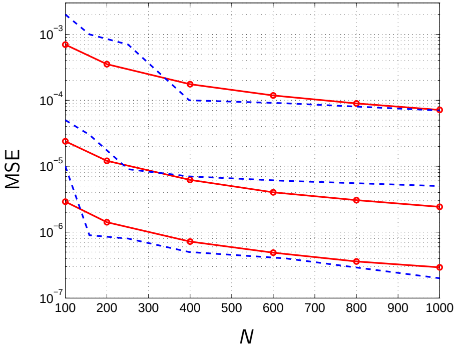


Estimation algorithm by Sascha Korl.

Does the algorithm perform well?

Results

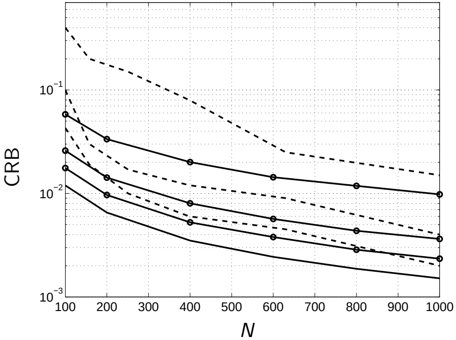
Results for σ_W^2 ($\sigma_U^2 = 0.1$; σ_W^2=0.001,0.01,0.1)



Estimation algorithm by Sascha Korl.

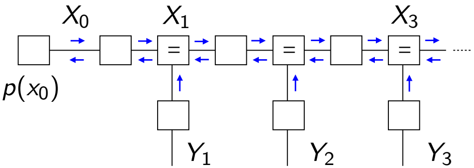
Results

Results for a ($\sigma_U^2 = 0.1$);



Standard CRB with unknown σ_U^2 and σ_W^2 (solid) for $\sigma_W^2 = 0.1/0.01/0.001$;
MSE of algorithm by S. Korl (dashed);
Standard CRB for a with known $\sigma_W^2 = 0$.

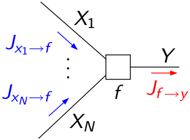
BCRB from information matrix of joint pdf



- The BCRBs of estimation in cycle-free graphical models can be computed efficiently by message passing.
- Messages are matrices.
- Messages are updated at each node according to specific update rules.
- The BCRBs are computed by combining those messages.

BCRB from information matrix of joint pdf (2)

Differentiable node function



$$J^{-1}_{f\to y}(Y) = \left(\begin{bmatrix} J_{x_1\to f}(X_1) + \mathrm{E}[-\Delta_{x_1}^{x_1}\log f] & \dots & \mathrm{E}[-\Delta_{x_1}^{x_N}\log f] & \mathrm{E}[-\Delta_{x_1}^{y}\log f] \\ \vdots & \dots & \dots & \vdots \\ \mathrm{E}[-\Delta_{x_N}^{x_1}\log f] & \dots & J_{x_N\to f}(X_N) + \mathrm{E}[-\Delta_{x_N}^{x_N}\log f] & \mathrm{E}[-\Delta_{x_N}^{y}\log f] \\ \mathrm{E}[-\Delta_{y}^{x_1}\log f] & \dots & \mathrm{E}[-\Delta_{y}^{x_N}\log f] & \mathrm{E}[-\Delta_{y}^{y}\log f] \end{bmatrix}^{-1}\right)_{N+1,N+1}$$

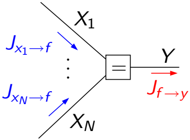
with $\Delta_{x_i}^{x_j} \triangleq \nabla_{x_i}\nabla_{x_j}^T$

Remarks

- Expectations $\mathrm{E}[\Delta_{x_i}^{x_j}\log f]$ supposed to be well-defined.
- They can easily be computed numerically.
- Rows and corresponding columns can be exchanged.

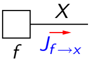
BCRB from information matrix of joint pdf (3)

Equality constraint node



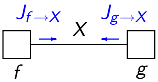
$J_{f \to y} = \sum_{i=1}^{N} J_{x_i \to f}$

Terminal node



$J_{f \to x} = -\mathrm{E}[\Delta_x^x \log f]$

PCRB



$J_{\mathrm{tot}} = J_{f \to X} + J_{g \to X}.$

Kernels from probability measures (2)

Probabilistic kernel

$$\kappa(\hat{y}_i, \hat{y}_j) \triangleq \sum_x p(\hat{y}_i, x|\hat{\theta})p(\hat{y}_j, x|\hat{\theta}),$$

with the parameters $\hat{\theta}$ are obtained from the whole data set $\mathcal{D}$, e.g., by ML-estimation:

$$\begin{aligned} \hat{\theta}^{\text{ML,tot}} &\triangleq \operatorname*{argmax}_\theta \prod_{i=1}^N p(\hat{y}_i|\theta) \\ &= \operatorname*{argmax}_\theta \prod_{i=1}^N \sum_x p(\hat{y}_i, x|\theta). \end{aligned}$$

Kernels from probability measures (3)

Product kernel [Jebara et al., 2004]

The product-kernel is computed as follows:

$$\kappa(\hat{y}_i, \hat{y}_j) \triangleq \sum_y p(y|\hat{y}_i)p(y|\hat{y}_j),$$

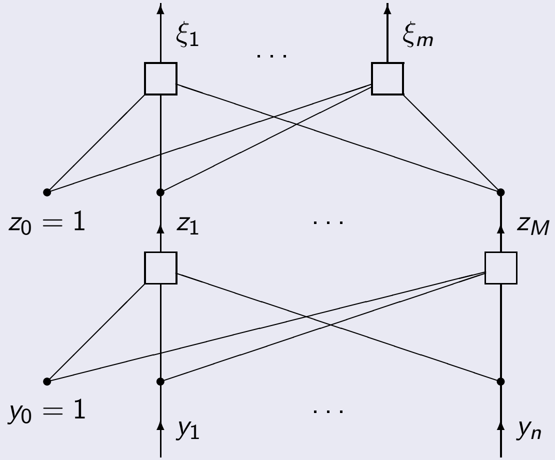
with

$$p(y|\hat{y}_i) \triangleq \sum_x p(y|x,\hat{\theta})p(x|\hat{\theta},\hat{y}_i),$$

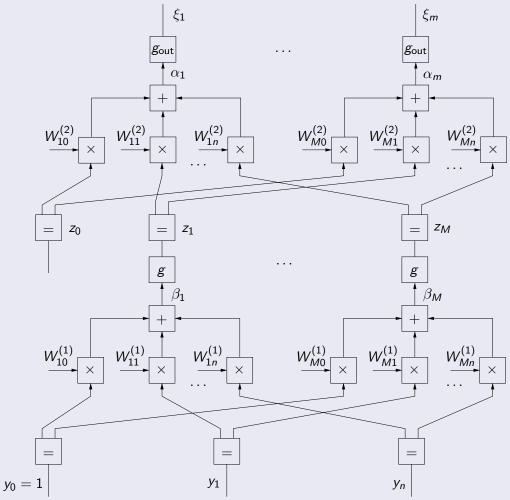
where the parameters $\hat{\theta}$ is estimated by means of the sample $\hat{y}_i$, e.g., by ML estimation:

$$\begin{aligned}\hat{\theta}^{\mathrm{ML}} &\triangleq \operatorname*{argmax}_\theta p(\hat{y}_i|\theta) \\ &= \operatorname*{argmax}_\theta \sum_x p(\hat{y}_i, x|\theta).\end{aligned}$$

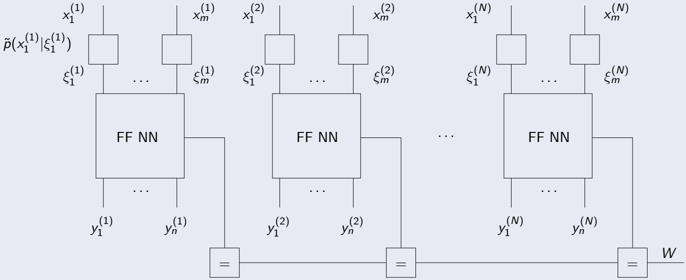
Feed-forward neural network



Feed-forward neural network



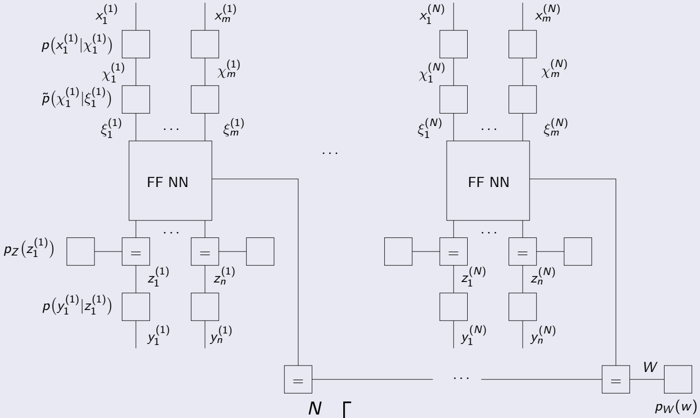
Feed-forward neural network (2)



$$p(x, \xi | y, w) \triangleq \prod_{\ell=1}^{N} f(\xi^{(\ell)}, y^{(\ell)}, w) \prod_{k=1}^{m} \tilde{p}(x_k^{(\ell)} | \xi_k^{(\ell)})$$

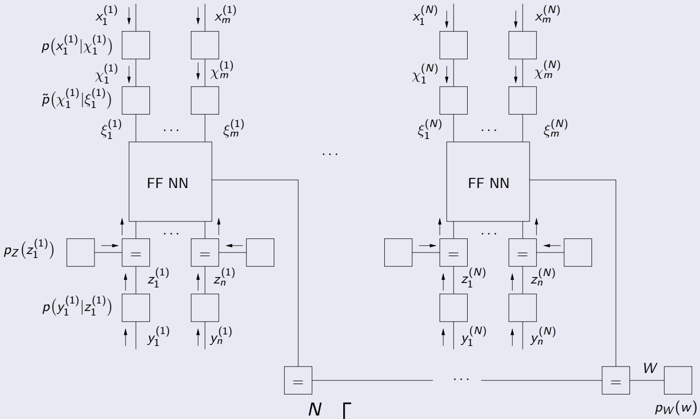
$$= \prod_{\ell=1}^{N} \delta(\xi^{(\ell)} - \xi(y^{(\ell)}, w)) \prod_{k=1}^{m} \tilde{p}(x_k^{(\ell)} | \xi_k^{(\ell)}).$$

Feed-forward neural network: additional nodes



$$p(x,\xi,z,y,w) \triangleq p_Z(z)p_W(w)\prod_{i=1}^{N}\Bigg[\delta(\xi^{(i)}-\xi(z^{(i)},w)) \cdot \left(\prod_{j=1}^{m}\tilde{p}(x_j^{(i)}|\chi_j^{(i)})p(\chi_j^{(i)}|\xi_j^{(i)})\right)\left(\prod_{j=1}^{n}p(y_j^{(i)}|z_j^{(i)})\right)\Bigg].$$

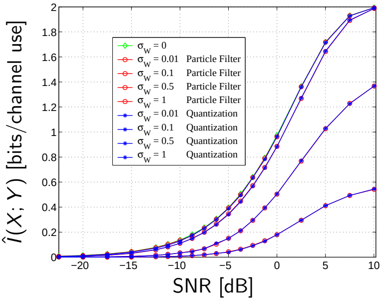
Feed-forward neural network: pre-processing

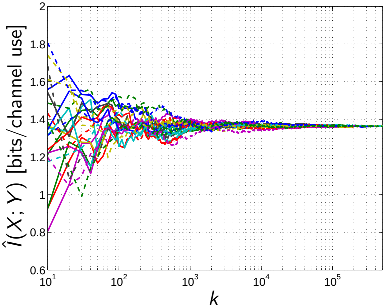


$$p(x,\xi,z,y,w) \triangleq p_Z(z)p_W(w)\prod_{i=1}^{N}\Bigg[\delta(\xi^{(i)}-\xi(z^{(i)},w)) \cdot \left(\prod_{j=1}^{m}\tilde{p}(x_j^{(i)}|\chi_j^{(i)})p(\chi_j^{(i)}|\xi_j^{(i)})\right)\left(\prod_{j=1}^{n}p(y_j^{(i)}|z_j^{(i)})\right)\Bigg].$$

Numerical results

Random-walk phase model with i.u.d. 4-PSK input symbols X





SNR $= 10$dB and $\sigma_W = 0.5$

Capacity of memoryless channel

Definition

$$C \triangleq \sup_{p(x)} \int_x \int_y p(x)p(y|x) \log \frac{p(y|x)}{p(y)} dxdy \triangleq \sup_{p(x)} I(X;Y)$$

with $p(y) \triangleq \int_x p(x)p(y|x)dx$.

Channel Coding Theorem

C $=$ highest rate at which information can be sent over the channel $p(y|x)$ with arbitrarily low P_e.

Two Blahut-Arimoto-type algorithms

Accelerated Blahut-Arimoto algorithm

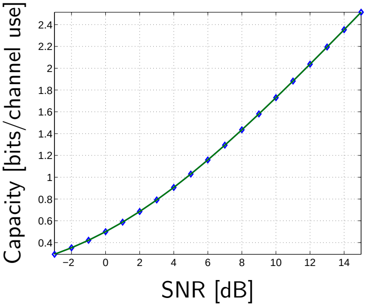
$$p^{(k)}(x) = \frac{1}{Z^{(k)}} p^{(k-1)(x)} \exp\left(\mu^{(k)} D\left(p(y|x) \| p^{(k-1)(y)}\right)\right)$$

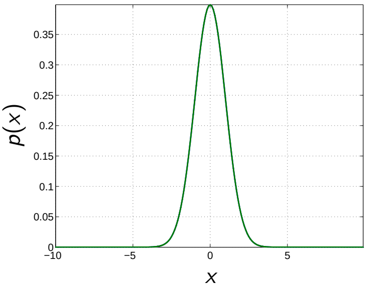
Natural-gradient based algorithm

$$p^{(k)(x)} = p^{(k-1)(x)} \left[1 + \mu^{(k)}\left(D\left(p(y|x) \| p^{(k-1)(y)}\right) - I^{(k-1)}\right)\right]$$

Results: Gaussian channel with $\mathrm{E}[X^2] \le P = 1$

$$C = \frac{1}{2}\log_2\left(1 + \frac{P}{\sigma_0^2}\right)$$

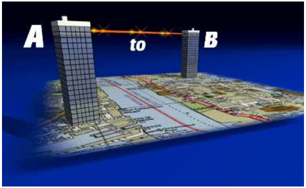


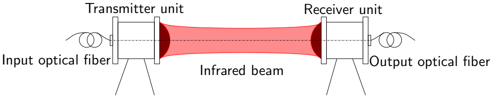


Results: Gaussian channel with $0 \leq X \leq 1$

Model for free-space optical communications channel

- Transmitter: light emitting diode (LED) or laser diode (LD)
- Signal modulated on optical intensity (ON/OFF keying)
- Direct line-of-sight path is dominant
- Noise source = ambient light
- Peak power constraint due to eye safety and potential thermal skin damage





Verification

Theorem (Karush-Kuhn-Tucker condition)

For memoryless channel with average-power constraint $E\left[X^2\right] \leq P$, $p(x)$ achieves capacity C iff there exists $\gamma \geq 0$ such that

$$\gamma(x^2 - P) + C - D\left(p(y|x) \| p(y)\right) \geq 0,$$

for all x, with equality for all $x \in supp[p(x)]$.

If no average-power constraint: $\gamma = 0$.